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Surveys in game theory and related topics

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TO STEF TIJS

FOREWORD

This book is dedicated to Stef Tijs, professor of Operations Research at the Mathematical Institute of the Catholic University of Nijmegen, The Netherlands. Many reasons are at the root of this book. The most compelling one is probably the fact that each of the authors received his/her introduction to game theory from Stef Tijs. His enthousiastic way of teaching was so convincing for them that they started research in game theory (or a related topic). His important role in their research programs can best be explained by numbers. For 9 of the 12 authors, Stef Tijs was advisor of their M.S.D. thesis. For 10 of them he was/is advisor of their Ph.D. thesis as well. Game theoretical research in the Netherlands can be compared with a star with spurs. The inner star is Stef Tijs. The spurs are his students, fed with his ideas as a base for their own research.
A second main reason for the dedication of this book to Stef Tijs is due to his leading role in the development of game theory in the Netherlands. Game theory in the Netherlands started with the Ph.D. thesis of Stef Tijs in 1975, "Semi infinite and infinite matrix games and bimatrix games". From that moment on he started building a game theory school. Step by step each subfield of game theory was covered by one of his students. On a national as well as on an international scale he took care of the establishment of the vital scientific contacts. Once a year he organizes an International Game Theory Day. Many famous game theoreticians visited Nijmegen; among them Shapley, Selten, Maschler, Schmeidler, Raghavan, Parthasarathy and Filar. To keep the Dutch game theoreticians in touch with each other, he organizes a seminar on game theory 4 to 6 times a year. At these seminars young researchers have the opportunity to speak about their current work.

Concerning Stef's own scientific contributions to the development of game theory : since 1976 he has published more than 60 outstanding papers in international journals.

Summarizing, it may be said, without overstatement, that game theory in the Netherlands owes its present state to Stef Tijs and it is this fact that the authors wish to memorize and emphasize with this book. The fact

that the appearance of this book coincides with Stef's 50-th birthday, August 31, 1987, is no coincidence at all.

For the technical realization of this book we are indebted to the Rijksuniversiteit Limburg, particularly to the Economic Department for granting us secretarial facilities.
Yolanda Paulissen typed the manuscript and took care of the layout.
She had a difficult task since she worked under considerable time pressure.
Yolanda, however, performed her job so extremely well, that this book had never become what it is without her persistent devotion and excellent typewriting.
Finally we wish to express our gratitude to the Centre for Mathematics and Computer Science at Amsterdam for publishing this book.

Hans Peters
Koos Vrieze

Rijksuniversiteit Limburg
August, 1987.

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INTRODUCTION

This book consists of surveys in game theory. Each author was asked to provide a survey on his/her specific topic within the field of game theory (or related topics), eventually supplemented with recent own results. Thus this book serves two goals. On the one hand, on a large number of topics, the reader has the opportunity to get a review of the state of the art of that topic. On the other hand, since recent developments are mentioned and since on several places proof methods are indicated, the reader may become familiar with the present way of thinking in game theory.

Game theory is a mathematical theory which deals with decision situations in which several persons with diverging preferences are involved. The foundation of game theory was laid by John von Neumann in 1928 ("Zur Theorie der Gesellschaftsspiele", 1928, Math. Annalen 100, pp.295-320). However the theory received widespread attention only after the publication of the fundamental book of Von Neumann and Morgenstern ("Theory of Games and Economic Behavior",1944, John Wiley & Sons, New York). Nowadays it is understood that game theory is able to provide indispensable tools for mathematical models and notions of many disciplines like economy, the social sciences, the political sciences and sociobiology.

Traditionally, game theory has been divided into two classes : cooperative games and non-cooperative games. This division can also be traced in this book. Chapters I through V are concerned with non-cooperative game theory, while the chapters VII through XII deal with cooperative game theory. The remaining two chapters (VI and XIII) contribute to the related topics.

Chapter VI is based on the observation that the same mathematical structure is underlying many problems in decision making under uncertainty and in game theory. It is argued that many theorems can be interchanged for these two fields merely by interchanging "state of nature" and "player".

In chapter XIII some social choice problems are treated. In the theory of social choice the strategic aspects of game theory is partially vanished. Individual preferences over the alternatives, among which a group of individuals has to decide, lead to decision procedures which determine the collective choice of preference. The emphasis is laid on the "impossibility" of procedures which, at first sight, are socially defendable and have appreciable properties.

In non-cooperative game theory no binding agreements between the players are allowed. Then solutions of such games have to be self-enforcing in the sense that, once it is agreed upon, nobody has an incentive to deviate. This point of departure leads to the Nash-equilibrium as solution concept for non-cooperative games, i.e. a strategy combination with the property that no player can gain by unilaterally deviating from it. Even games with a finite number of pure strategies for the players may have several or an infinite number of equilibrium points. Hence, in the literature, several refinements of the Nash-equilibrium are proposed. In chapter I the most important ones are introduced in an intuitive way for games in extensive form. Also the relation with the associated normal form game are explained. Chapter II focusses on games in normal form. For the two player case the set of equilibria are analysed in detail. Several refinement concepts are considered and interrelated. Chapter III presents a survey on two player games with incomplete information. As well the zero-sum case as the nonzero-sum case are treated. Many problems remain open in this field. A recent International Conference on Game Theory (Columbus, Ohio, June 18-24, 1987) showed that this field may rejoice in a wide interest at the moment.
Chapters IV and V deal with stochastic games. A stochastic game is a dynamic system, where at discrete time epoques the players have to make a decision, resulting in immediate rewards and in a Markovian stochastic movement of the system. In chapter IV a survey on zero-sum stochastic games is given. Three criteria are considered : discounted -, average - and total reward stochastic games. In chapter V nonzero-sum stochastic games are treated.

In cooperative game theory binding agreements between the players play the crucial role. This is reflected by the way cooperative games are defined, namely in characteristic function form, i.e. for each possible coalition (subset of players) a real number is given, being the worth of the coalition. In games with transferable utility (chapters VII through XI) the question is, how to divide the worth of the grand coalition.

Since the before mentioned book of Von Neumann and Morgenstern, several solution concepts are proposed. In chapter VII the probably most important one is analysed in detail : the core, i.e. the set of efficient divisions which are individual rational and group rational. In chapter VIII the τ-vector and her properties are considered. This solution concept is introduced by Tijs (S.H. Tijs, 1987, "An axiomatization of the τ-value", Math. Social Sciences 13). It appears to have several nice properties, making it suitable for application in many economic situations.
In chapter IX the subclass of superadditive games are dealt with. For these games formation of coalitions makes sense, as the worth of the union of two disjoint coalitions of players is at least as large as the sum of the worth's of the separable coalitions.
In the chapters X and XI the theory of cooperative games is applied to multiperson combinatorial optimization problems. The worth of coalitions results by solving certain combined combinatorial problems. In chapter X for seven classic combinatorial problems the multiperson version is formulated. It is shown, for instance, that for five of them the core is nonempty. In chapter XI linear optimization games are considered. Special attention is paid to problems with discrete action spaces. An attempt is made to unify the existing methods.
Finally, chapter XII is concerned with bargaining theory, which belongs to the theory of cooperative games without side-payment.
In games without side-payment a set of feasible outcomes is given together with a status quo point, being the outcome if the players do not succeed in agreeing on some feasible outcome.
A survey is presented with respect to the nonsymmetric Nash bargaining solutions with emphasis on different axiomatic characterizations and the economic meanings of these axioms.

CONTENTS

CHAPTER I

EQUILIBRIA IN NONCOOPERATIVE GAMES

by Eric van Damme

ABSTRACT

This paper surveys the main equilibrium concepts that have been proposed for (strictly) noncooperative games, i.e. games without communication in which no binding agreements nor commitments can be made. The most important properties of these concepts are presented and illustrated by means of simple examples. Emphasis is on concepts that are especially relevant for extensive form games.

The author should like to thank Stef Tijs for introducing him to Game Theory and for his encouragement at critical stages of the development. This paper owes a considerable intellectual debt to work of Harsanyi and Selten and Kohlberg and Mertens. Financial support from the Deutsche Forschungsgemeinschaft through the Sonderforschungsbereich 303 is gratefully acknowledged.

1. INTRODUCTION

Game Theory can be defined as the study of mathematical models of conflict and cooperation between rational individuals. It is a normative theory of which the aim can be described as 'We wish to find the mathematically complete principles which define "rational behavior" for the participants in a social economy and to derive from them the general characteristics of that behavior' (Von Neumann and Morgenstern (1947,p.31)). Von Neumann and Morgenstern insist that the principles ought to be perfectly general, but that we may be satisfied if we can find solutions only in special cases. (After more than 40 years it must be admitted that solutions have been found only in very special cases (such as 2-person zerosum games and games with perfect information) and that even in these cases the proposed solutions have not been universally accepted). A solution is defined as a set of rules which tell each player how to behave in every situation that may conceivably arise and Von Neumann and Morgenstern advocate the indirect method to find such a solution (or theory). This method consists in 'imagining that we have a satisfactory theory of a certain desired type, trying to picture the consequences of this imaginary intellectual situation and then in drawing conclusions from this as to what the hypothetical theory must be like in detail. If this process is applied successfully it may narrow down the possibilities for the hypothetical theory of the type in question to such an extent that only one possibility is left, - i.e. that the theory is determined' (Von Neumann and Morgenstern (1947,p.147)). Let us assume that this indirect method is successfull and that it has produced a unique, absolutely convincing, solution for a certain game. If this theory is common knowledge, then each player knows the rule recommended to each opponent, and a player will play according to the theory only if his recommended strategy maximizes his payoff when played against the strategies of the others. Hence, in game theoretic terminology, a self-confirming theory must prescribe a Nash equilibrium.

Nash's (1951) equilibrium concept is the most fundamental and most important one in Game Theory. It is relevant for (strictly) noncooperative games, i.e. games in which there are no possibilities for communication, correlation or (pre-)commitment except for the ones that are explicitly allowed by the rules of the game. It should be stressed that a noncooperative game does not prohibit cooperation, it just requires that all possibilities for cooperation be explicitly modelled. It will be clear that, therefore, a noncooperative model may have to be very detailed and it may be more convenient to omit, for example, possibilities for communication from the game and to build them into the solution concept instead (in this case one obtains Aumann's (1972) concept of correlated equilibria). One may even wish to build commitment possibilities into the solution concept (one then enters the realm of cooperative game theory). However, it should always be possible to return to fully noncooperative modelling and to justify the solution concept by means of Nash equilibria of the underlying game. In this paper, attention will be confined to strictly noncooperative games, hence, to Nash equilibria.

A noncooperative game can be represented either in extensive form (which describes the actual evolution of the play in every detail) or in normal form (in which attention is just on outcomes). Both models are formally introduced in section 2 and in the sections 2 and 6. We discuss the issue of whether these 2 representations are equivalent. (The main problem is whether the normal form, which condenses all decision making before the actual play, involves self-commitment or not). In section 3 we try to clarify some confusion concerning the concept of Nash equilibria. Von Neumann and Morgenstern's indirect argument discussed above establishes only that being a Nash equilibrium is necessary for being a rational solution of a noncooperative game, it does not establish sufficiency and indeed not all Nash equilibria are satisfactory. Specifically, not all Nash equilibria are self-enforcing, i.e. not all equilibria have the property that, when recommended, no player has an incentive to deviate. In the sections 4 and 5, the drawbacks of the Nash concepts are discussed, we try to identify which criteria self-enforcing equilibria should satisfy and introduce some more refined equilibrium concepts based on such criteria. Finally, in section 6 it is discussed whether a meaningful equilibrium concept for extensive form games can be based on the normal form.

2. NONCOOPERATIVE GAME MODELS

A noncooperative game can be represented in extensive form, in normal form or in agent normal form. This section provides the formal definition of these forms as well as of the basic concepts associated with them. We open with the extensive form, which is the most detailed model and which closely follows the actual evolution of the play, i.e. this model exactly specifies 'who moves when', 'who knows what' and 'what are the consequences of which'. Throughout, attention is confined to finite games. If the discussion appears too terse, the reader may turn to Selten (1976) or Van Damme (1983) for a more leisurely presentation.

An *n-person extensive form game* is a sixtuple $\Gamma = (K,h,P,p,U,C)$ of which the constituents are as follows :

(i) K is a finite *directed tree* with a distinguished vertex (the origin). The interpretation is that the game starts at the origin and moves from vertex to vertex until an endpoint is reached. The nonterminal vertices are decision points, a decision determines an edge of the tree, hence, a vertex where the game continues. We write $x < y$ if vertex y comes after x in the tree and E_x denotes the set of edges originating at x, i.e. if $e \in E_x$ then e connects x to y where y comes immediately after x.

(ii) h is the *payoff function*, it specifies for each player i and each endpoint z the payoff $h_i(z)$ that i receives when z is reached. (No payoffs are cashed during the game).

(iii) P is the *player labelling*, i.e. P labels each decision point with the player who has to move at this point. We write P_i for the points where player i has to move, hence $P_i = \{x; P(x) = i\}$. Some points may be labeled with 0, the interpretation being that in this case a chance move (move of nature) is performed.

(iv) p specifies the *probabilities of the chance moves*, hence, p_x is a probability distribution on E_x for every $x \in P_0$.

(v) $U = (U_1,...,U_n)$ where U_i is a partition of P_i into *information sets* of player i. The interpretation is that, if $x \in P_i$ is reached by the play, then player i only knows that the element $u \in U_i$ that contains x is reached. Each partition U_i must satisfy the following condition : If $u \in U_i$ and $x,y \in u$, then $|E_x| = |E_y|$ and $\neg(x < y)$. In words, nodes at the same information sets have the same number of edges (see (vi)) and each play intersects each information set at most once.

(vi) $C = (C_u)_u$ specifies for each information set u ($u \in \bigcup_i U_i$) the *choices* available at u. Formally, C_u is a partition of $\bigcup_{x \in u} E_x$ such that for each $c \in C_u$ and each $x \in u$, the choice c contains exactly 1 alternative from E_x. The interpretation is that, if player i takes the choice c at u and if the play actually is at $x \in u$, then the play moves to the node after $c \cap E_x$.

As an illustration, consider the game of Fig. 1a. Player 1 has to move at the origin where he has to choose between A and B. (Note that the origin always constitutes a singleton information set). If 1 chooses B, the game terminates with 1 receiving 2 and player 2 getting 4. If 1 chooses A, he has to choose again, this time between L and R. Player 2 has to move after player 1 has chosen for the second time and, as indicated by the dotted line connecting the decision points of player 2, this player does not know whether 1 has chosen L or R, hence, the 2 decision points of player 2 constitute an information set. Player 2 has 2 choices, l and r, where l corresponds to the left branches that originate in 2's decision points. The game terminates after player 2 has moved. Each endpoint determines a pair of payoffs with the understanding that the first number always is the payoff to player 1.

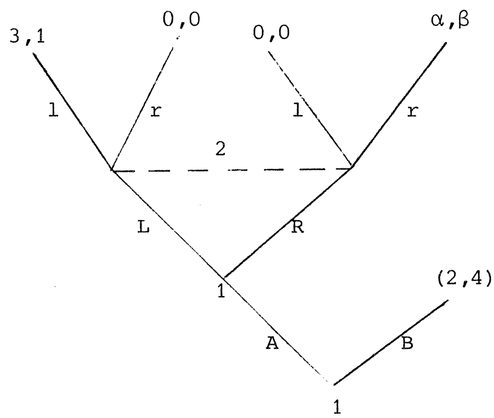

Fig. 1a

	l	r
AL	3,1	0,0
AR	0,0	α,β
BL	2,4	2,4
BR	2,4	2,4

Fig. 1b

Next, let us turn to the question of how to play an extensive form game, i.e. to the concept of strategy. A *behavior strategy* σ_i of player i specifies a probability distribution on C_u for every information set u of this player. If σ_i actually specifies a choice (i.e. a degenerate distribution) for each $u \in U_i$, then σ_i is called a *pure strategy*. A *mixed strategy* is a probability distribution over the (finite) set of pure strategies. Note that mixed strategies correspond to prior randomization whereas behavior strategies involve only local randomization. Clearly, every mixed strategy induces a behavior strategy, but the converse need not hold since a mixed strategy allows correlation between actions at different information sets. If the game has *perfect recall* so that every information set always discloses what one has known or done in the past, then there is no need for such correlation and in this case the restriction to behavior strategies is justified (Kuhn (1953), Formally Γ is said to have perfect recall if for any i, all information sets $u,v \in U_i$ and all choices $c \in C_u$: if some point in v comes after c (i.e. after some edge in c) then all points in v come after c). Throughout, attention will be confined to games with perfect recall.

A *strategy combination* σ is an n-tuple of strategies, one for each player. When each player has chosen his strategy (and decides to stick to it), then an outside observer can determine the outcome of the game. Specifically, each strategy combination σ uniquely determines a probability distribution $\mathbb{P}(\sigma)$ over the endpoints of the tree, which will be called the outcome of σ. Player i's *expected payoff* resulting from σ is the expectation of h_i with respect to $\mathbb{P}(\sigma)$, hence $H_i(\sigma) = \Sigma_z h_i(z)\, \mathbb{P}(z \mid \sigma)$. Two strategies σ_i' and σ_i'' will be said to be *equivalent* (resp.payoff equivalent) if $\mathbb{P}(\sigma\backslash\sigma_i') = \mathbb{P}(\sigma\backslash\sigma_i'')$ for all σ (resp. $H_j(\sigma\backslash\sigma_i') = H_j(\sigma\backslash\sigma_i'')$ for all σ and j) where $\sigma\backslash\sigma_i'$ is short hand notation for $(\sigma_1,\ldots,\sigma_{i-1},\sigma_i',\sigma_{i+1},\ldots,\sigma_n)$. In Fig. 1a the strategies BL and BR are equivalent. The strategy combination σ is said to be (payoff) equivalent to σ' if σ_i is (payoff) equivalent to σ_i' for all i.

We now turn to games in normal (or strategic) form. An n-person *normal form game* is a 2n-tuple $(S_1,\ldots,S_n,H_1,\ldots,H_n)$ where S_i is a finite, nonempty set (of pure strategies of player i) and $H_i : S \to \mathbb{R}$ is this player's payoff function ($S = X_i S_i$) with the interpretation that players choose their strategies simultaneously and independently. Such a game can be viewed as a special kind of extensive game, but the above discussion makes it clear that we can also associate a normal form game to each game Γ that is originally given in extensive form (just let S_i be player i's pure strategies in Γ). Fig. 1b displays the normal form of Fig. 1a with the usual convention that player 1 chooses a row. This example already shows that usually the normal form contains a lot of redundancy, for example, payoff equivalent strategies just appear as duplications. The game that results after all such duplications have been removed will be called the *semi reduced normal form*. If one also eliminates pure strategies that are duplications of a mixture of other pure strategies, the *reduced normal form* results.

If one replaces an extensive game by its normal form, one condenses all decision making into 1 stage without actually limiting the freedom of action that a player has, so one may argue (as in Von Neumann and Morgenstern (1947,p.85) that the 2 representations are fully equivalent. However, note that the normal form assumes that there is full coordination between choices of the same player at different information sets and this need not be the case in the extensive form. For example, a player may be a corporation which has, at different information sets, different agents (all with the corporation's payoff function) acting on behalf of it. If the agents receive coordinated instructions, the normal form is appropriate, but if they act independently, they should be treated as seperate, active players. In this case, not the normal form is relevant, but rather the *agent normal form*, i.e. the normal form game $\langle C_u, H_u \rangle_u$ where $u \in \cup_i U_i$ and $H_u = H_i$ if $u \in U_i$ (see Selten (1976)). Hence, before normalizing a game, one should first determine which agents are active and which are dummies. One may argue (philosophically) that 'person i at information set u' is always different from 'person i at information set v' if $u \neq v$ so that the agent normal form is always appropriate and this brings us to the second problem associated with the equivalence asserted above : The normal form seems to imply immediate commitment, since as soon as one has chosen a strategy one has to stick to it whereas deviations are possible during an extensive game. This author does not consider this issue problematic for, if one would deviate from one's strategy in the extensive game, one would forsee this at the beginning of the game and also deviate in the normal form. Hence, we will take the point of view that normal form strategies are relevant (at least for players which are single individuals), but we will return to the issue in section 6. Let us conclude by remarking that for games with incomplete information, this view is in agreement with that of Harsanyi (1968) : In an incomplete information game, different types of the same player act fully independent so that the agent normal form is appropriate and not the normal form.

3. NASH EQUILIBRIA

Let Γ be a game in which player i's payoff function is H_i. The strategy σ_i' is said to be a *best reply* against σ if

$$H_i(\sigma\backslash\sigma_i') = \max_{\sigma_i''} H_i(\sigma\backslash\sigma_i'').$$

σ' is a best reply against σ if the above equality holds for all i. If σ is a best reply against itself, then σ is called a *Nash equilibrium*. Hence, σ is a Nash equilibrium iff no player can gain by deviating unilaterally from σ. Note that the Nash concept excludes reactions from j to deviations from i. This is without loss of generality since all such possible reactions are already captured by the rules of our (noncooperative) game. Also note that, if Γ is an extensive game with perfect recall, then σ is a Nash equilibrium of Γ in behavior strategies if and only if σ is payoff equivalent to a mixed strategy equilibrium of the normal form of Γ. Nash (1950) has shown that any (finite) game has a Nash equilibrium. Also the structure of the set of equilibria has been investigated in some detail (see e.g. Jansen (1981) for the 2-person case). For later reference, we collect some properties in Proposition 1. (We say that property P holds for generic extensive games if, for any n-person extensive structure (K,.,P,p,U,C), the set of payoff functions $h \in \mathbb{R}^{nZ}$ for which P fails to hold is contained in a closed set with Lebesgue measure zero (Z denotes the endpoints of K)).

<u>Proposition 1</u>

(i) (Nash (1950)). Every game has a Nash equilibrium.

(ii) (Kohlberg and Mertens (1986)). The set of Nash equilibria consists of finitely many connected components.

(iii) (Kreps and Wilson (1982)). Generic extensive form games have finitely many Nash equilibrium outcomes so that all equilibria in the same component yield the same outcome.

(iv) (Harsanyi (1973a)). Generic normal form games have an odd number of equilibria which vary differentiably with the payoffs of the game.

This proposition may be illustrated by means of the game of Fig. 1. Let $0 < \alpha < 2$ and $\beta = 1$. Then there are 2 components, viz $\{(AL,l)\}$ and $\{(\sigma_1,\sigma_2)\}$ where σ_1 is any mixture of BL and BR and σ_2 chooses l with probability at most 2/3. The first component yields the outcome (3,1), the second yields (2,4). (Note that this game already shows that 'finite' cannot be replaced by 'odd' in Proposition 1(iii)). As the normal form from Fig. 1b is nongeneric, (iv) does not apply. However, the subgame starting at the second decision node of player 1 yields a generic normal form and this has Nash equilibria (L,l), (R,r) and (σ_1,σ_2) where σ_1 chooses both L and R with probability 1/2 and σ_2 chooses l with probability $\alpha(3+\alpha)^{-1}$, hence, the equilibria indeed vary differentiably with α.

In the Introduction we already justified the Nash concept and explained why this concept is fundamental to noncooperative game theory. However, this (indirect) justification does not imply that unaided players will necessarily choose equilibrium strategies, nor have we specified an equilibriating (learning) process by means of which they could reach equilibrium. Hence, Nash equilibria may be irrelevant from a positive point of view and it is not clear why one should actually play a Nash equilibrium. Of course, if a game has a single Nash equilibrium, if this is strict (i.e. each player will loose by deviating) and if one expects the opponents to play the equilibrium, then one should also play one's equilibrium strategy, but why should one expect the others to play the equilibrium ? Following Schelling (1960) one can argue that in this case only the equilibrium is a focal point which serves to coordinate expectations. (This should especially be the case if all players are familiar with the argument from the Introduction).

However, even in this case, the Nash concept has been challenged in Bernheim (1984) and Pearce (1984). Consider Fig. 2a which has a unique equilibrium at (R_1,R_2). Bernheim and Pearce argue that one should not necessarily expect this equilibrium to be played as players might analyse the game with inconsistent beliefs (player 1 (2) might firmly believe that 2 (1) is going to play L_2 (L_1) in which case the outcome will be (L_1,M_2); more generally, all outcomes are 'rationalizable' in this game). Of course, one must ask where such inconsistent prior beliefs might come from given that players have common information about the game. In fact, Aumann (1987) strongly argues in favor of common priors and Bernheim (1986) has shown that common priors do indeed lead to Nash outcomes rather than just rationalizable outcomes.

	L_2	M_2	R_2
L_1	5,0	0,5	0,3
M_1	0,5	5,0	0,3
R_1	3,0	3,0	1,1

Fig. 2a

	L_1	R_1
L_2	8,8	0,6
R_2	6,0	6,6

Fig. 2b

	L_1	R_1
L_2	1,-1	-1,0
R_2	0,2	0,0

Fig. 2c

Schelling's focal point argument looses much of its force when there are multiple equilibria as in Fig. 2b. In this game, there are 2 strict pure equilibria, viz (L_1,L_2) and (R_1,R_2) and the former Pareto dominates the latter so that one might say that (L_1,L_2) is more focal. On the other hand, R_i is a much safer strategy as it guarantees a payoff of 6 no matter which action the opponent chooses. So should one be surprised if players fail to coordinate their choices and the outcome turns out to be (L_1,R_2) ? To this point one can respond by saying that Nash equilibrium is best viewed as a 'presolution concept' rather than as a solution concept, since being a Nash equilibrium is only a necessary condition, not a sufficient condition for being the rational solution of a game. Hence, as soon as game theorists have found out which of the above 2 arguments is more prominent, they will be able to ascribe a unique solution to the game of Fig. 2b and Schelling's argument again applies. Consequently, it seems important to have an equilibrium selection theory. One such theory has been developed in Harsanyi and Selten (1987), but lack of space prevents a discussion here.

To guarantee existence of equilibria it is essential that one allows randomization and equilibria in mixed strategies have always given rise to puzzlement and misunderstanding. After all, the idea of basing important decisions on a toss of a coin is hard to accept. Furthermore, such equilibria seem difficult to interpret as randomization is not necessary for one's own purposes but rather to keep others from deviating. Hence, on the face of it, mixed strategy equilibria seem unstable as a player always has alternative (pure) best replies available. These issues can be illustrated by means of the game of Fig. 2c. This game has ((2/3,1/3), (1/2,1/2)) as its unique equilibrium, i.e. player 1 chooses his first strategy with probability 2/3, player 2 randomizes equally. The equilibrium payoff is zero to both players, but the equilibrium strategies do not guarantee this payoff (one might get less if the other deviates). Hence, equilibrium strategies need not have maximin properties, and Aumann and Maschler (1972), therefore, suggest that player i might prefer to play R_i. However, (R_1,R_2) is not an equilibrium so that player 2 will deviate if he expects 1 to follow the Aumann/Maschler suggestion. Also note that, in equilibrium, a player does not play for himself but against his opponent : player i randomizes to make j ($j \neq i$) indifferent. The latter is made even clearer if one changes player 1 payoffs but leaves the best reply structure unchanged, i.e. L_1 (resp. R_1) remains a best reply against L_2 (resp, R_2). Then player 1's equilibrium strategy remains constant, but 2's equilibrium strategy varies, hence, for a fixed best reply structure, a player's equilibrium strategy depends only on his opponent's payoffs, which all seems quite paradoxical.

Harsanyi (1973a) has argued that these paradoxes and seeming instabilities are caused by the fact that our model is too much idealized, i.e. it fails to incorporate the actual uncertainty that each player has about his opponents' payoffs, and that the difficulties disappear in a fully specified model. To illustrate Harsanyi's approach, consider Fig. 2c but now assume that player i does not know j's payoff associated to (R_1,R_2), each player knows his own payoff, however. (For simplicity we assume that only the payoff to (R_1,R_2) is uncertain). Specifically, suppose that i considers j's payoff ($i \neq j \in \{1,2\}$) associated to (R_1,R_2) to be a random variable εX_j where X_j has a *nonatomic* distribution F_j and ε is close to zero (X_1 and X_2 are assumed independent). As player 1 does not know the realization of X_2, he does not know whether player 2 will play L_2 or R_2. Let g be the probability player 1 assigns to 2 playing L_2. Then 1 will play L_1 for all realizations x_1 of X_1 with $x_1 \leqq (2g-1)/(1-g)\varepsilon$. Note that 2 is indifferent with probability 0 so that his best response is almost always unique and pure. Consequently, player 2 will assign a probability $p = F_1((2g-1)/(1-g)\varepsilon)$ to the event that 1 plays L_1 and this player will play L_2 for any realization x_2 with $x_2 \leqq (3p-2)/(1-p)\varepsilon$, hence, the ex ante probability that 2 plays L_2 is $F_2((3p-2)/(1-p)\varepsilon)$. Clearly, to have an equilibrium in this game, the beliefs of the players should be consistent, i.e.

$$g = F_2((3p-2)/(1-p)\varepsilon) \text{ and } p = F_1((2g-1)/(1-g)\varepsilon).$$

These equations have a solution (Brouwer's Fixed Point Theorem) and as ε tends to zero, all solutions converge to the mixed equilibrium of the game of Fig. 2c. Hence, the mixed equilibrium of the original game can be viewed as the beliefs associated with the pure equilibrium of the more detailed model in which the incomplete information is taken into account. Harsanyi (1973a) has shown that this interpretation holds for all mixed equilibria of generic normal form games so that the above mentioned instabilities are indeed seeming instabilities.(It should be remarked that this issue has not been completely settled for extensive games, and that, if one accepts Harsanyi's interpretation, it is not clear whether a normal form game is equivalent to its reduced normal form).

Above we argued that only Nash equilibria can be self-enforcing, but we did not discuss whether all equilibria have this property, i.e. whether for any Nash equilibrium, players will always follow the recommendation to play this equilibrium. The answer to this second question is negative as the Nash concept suffers from at least 3 drawbacks :

(i) Nash equilibria need not be sequentially rational, i.e. in extensive form games, they may prescribe suboptimal behavior at unreached information sets (so that a player will deviate when such a set is unexpectedly reached),

(ii) Nash equilibria need not be robust with respect to perturbations in the data of the game (so that players may deviate as soon as they entertain the slightest doubt about these data), and

(iii) Nash equilibria are not 'independent of irrelevant alternatives', i.e. they need not persist when unreasonable (e.g. dominated) actions are eliminated from the game.

In the next sections we will study these issues more deeply and discuss some more refined equilibrium concepts that have been introduced to overcome these drawbacks.

4. EQUILIBRIA IN EXTENSIVE FORM GAMES

The following simple example may demonstrate that Nash equilibria need not be sequentially rational.

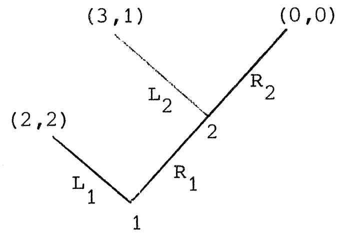

Fig. 3

If player 2's decision node is reached, player 2 will choose L_2, hence, backwards induction suggest that the solution of this game should be (R_1,L_2). This indeed is a Nash equilibrium, however, there is a second equilibrium, viz (L_1,R_2). In the latter equilibrium, 2 threatens that he will choose R_2 if he is reached and it is an equilibrium only since 2's decision point is not reached so that 2 does not have to execute his threat. Of course, a rational player 1 should foresee that 2's threat is empty and that 2 will choose L_2 if he is actually reached. Hence, (L_1,R_2) is not self-enforcing as player 1 has an incentive to deviate.

The above argument generalizes to all games with perfect information (i.e. games in which all information sets are singletons) and it leads to the conclusion that only equilibria that can be found by dynamic programming (*backwards induction*) are self-enforcing. Note that for generic perfect information games no player is ever indifferent between 2 choices so that almost always we obtain a unique solution. Several comments are in order concerning this solution :

(i) In experiments one finds that empty threats are carried out and that the game theoretic solution has nearly no predictive power (see e.g. Güth et.al. (1982), Güth and Tietz (1987)).

(ii) Fig. 3 can be interpreted as a game in which first an entrant (player 1) decides about whether to enter a certain market or not and next the existing monopolist decides about whether or not he should fight this entry (R_2 = fight entry). Suppose the same monopolist plays this game in n towns (n finite) against n different players such that player k is informed about the outcome in any town l with $l < k$. Intuition suggests that the monopolist will fight entry early on in the game to build a reputation for being tough, but backwards induction shows that this strategy is incredible : player 2 should give in (choose L_2) in all towns. This phenomenon is known as the *chain store paradox* and was first described in Selten (1978).

(iii) Backwards induction meets with philosophical problems. Namely, the induction process assumes that, given any point in the tree, from then on all players will play 'rationally' whereas the point itself might have been reached only as the consequence of 'irrational' behavior. Hence experience may refute the rationality assumption. To be specific, in the above chain store game the 'rational' solution tells player 2 always to choose L_2. Now suppose you are player 1 in town 25 and you observed that 2 choose R_2 on all 24 previous occasions. Do you still decide to choose R_1 ? (See Selten (1978) and Binmore (1985) for more on this issue).

(iv) Adding a little bit of imperfect information prevents the backwards induction to operate and this may in fact change the solution completely. For example, suppose that in the chain store game, there is a small probability that 2 actually prefers R_2 to L_2. Then, after fought entry, entrants will consider it more likely that the monopolist actually likes to fight and in this case the above discussed intuitive (agressive) strategy becomes credible. In fact, no matter how small the ex ante probability is that the monopolist likes to fight, the intuitive strategy becomes the unique solution as long as the number of towns is sufficiently large (see Kreps and Wilson (1982b)).

Despite the philosophical problem discussed above, game theorists have accepted the backwards induction principle and issue (iv) has motivated them to try to extend it to games with imperfect information. Selten (1965) proposed the concept of subgame perfect equilibria, i.e. of Nash equilibria that induce an equilibrium in every subgame. In Fig. 5 only (R_1,L_2) is subgame perfect as there is a subgame starting with the move of player 2. In Fig. 1a, if $\alpha < 0$, then (L,l) is the unique equilibrium of the subgame, so that only (AL,l) is subgame perfect. (In this case, (BL,r) is an imperfect equilibrium. Formally, if x is a decision point in an extensive game Γ then the restriction Γ_x of Γ to K_x (i.e. to the subtree starting of x) is a *subgame* if for every information set u of Γ either $u \subset K_x$ or $u \cap K_x = \emptyset$. The strategy combination σ is said to be a *subgame perfect equilibrium* if σ_x (i.e. the restriction of σ to Γ_x) is a Nash equilibrium of Γ_x for every subgame Γ_x. An equilibrium is said to be *subgame consistent* (Selten (1973)) if it prescribes the same equilibrium in any 2 subgames Γ_x and Γ_y that differ only in the history about how x (resp. y) is reached. (Note that subgame consistency generalizes the idea of Markov strategies). Proposition 1 (i) implies that, if σ_x is an equilibrium of the subgame Γ_x, then σ_x can be extended to an equilibrium of Γ (just replace Γ_x by its equilibrium value and take an equilibrium of the truncated game). The following Proposition is a trivial consequence of this observation.

<u>Proposition 2</u>

(i) Every game has a subgame perfect equilibrium, in fact, it has even a subgame consistent equilibrium.

(ii) Generic extensive games with perfect information have a unique subgame perfect equilibrium.

The concept of subgame perfectness has found many applications in economics, but space restrictions prevent a discussion at this point. (An application which has come to play a prominent role in economic theory is Rubinstein (1982), for some other applications, see Van Damme (1987)). Unfortunately, the subgame perfectness concept is not satisfactory from a theoretical point of view. Namely, suppose Fig. 3 is changed so that R_1 is duplicated, i.e. player 1 chooses simultaneously between L_1,R_1 and R_1', player 2 does not get to hear whether R_1 or R_1' was chosen, but the payoffs after R_1' are identical to those after R_1. This duplication does not change the strategic situation, but the new game has no proper subgames, so that in particular the equilibrium (L_1,R_2) is subgame perfect. To eliminate such drawbacks, Selten (1976) proposed the concept of perfect equilibria. This concept is based on the *'trembling hand' principle*, i.e. it is assumed that each player makes mistakes with an infinitesemal probability so that each choice will occur with small positive probability which makes every information set reachable and thereby eliminates empty threats and nonoptimal behavior. Formally, to be a *perfect equilibrium* it is required that at each information set each player's action is optimal not only against the equilibrium actions of the others but also against small perturbations thereof (hence a perfect equilibrium of the extensive form is just a perfect equilibrium of the agent normal form (see section 5)).

In the modified game of Fig. 3, if player 1 chooses R_1 and R_1' with positive (mistake) probability, then 2 has only L_2 as a best reply so that only the equilibria with payoff (3,1) are perfect.

Note that the perfectness concept establishes sequential rationality by requiring robustness. Such robustness may be difficult to verify, however, and this motivated Kreps and Wilson (1982a) to introduce the slightly modified concept of sequential equilibria. The idea underlying this concept is that at every information set a player will construct beliefs about in which point of the set he actually is and optimize, given these beliefs, against the opponents' strategies. Kreps/Wilson require that these beliefs be common and common knowledge and that they be consistent with the equilibrium strategies, which they formalize by insisting that beliefs can be explained by means of small deviations from the equilibrium. Formally, a *system of beliefs* assigns to each nonterminal node x a nonnegative number $\mu(x)$ such that a probability distribution results over each information set. By Bayes' rule, an interior behavior strategy combination σ (i.e. each choice occurs with strictly positive probability) induces a unique system of beliefs $\mu(\sigma)$ given by $\mu(x \mid \sigma) = \mathbb{P}(x \mid \sigma) / \mathbb{P}(u \mid \sigma)$ for $x \in u$. More generally, μ is said to be *consistent* with σ if $(\sigma,\mu) = \lim_k (\sigma^k, \mu(\sigma^k))$ for some sequence $(\sigma^k)_k$ of interior strategy combinations. For each information set u, the assessment (σ,μ) induces a probability distribution $\mathbb{P}^{\mu}_{u}(\sigma)$ on the terminal nodes of the tree and $H^{\mu}_{iu}(\sigma)$ denotes player i's expected payoff with respect to this distribution ($\mathbb{P}^{\mu}_{u}(\sigma)$ is obtained by first drawing a point in u according to μ and then playing the game as described by σ). Finally, σ is said to be a *sequential equilibrium* if there exists a system of beliefs μ that is consistent with σ such that σ is a Nash equilibrium of the agent normal form game in which agent u of player i has payoff function H^{μ}_{iu}.

It is not difficult to show that a sequential equilibrium is subgame perfect. The converse is not true, as in the modification of Fig. 3 discussed above, no matter which beliefs player 2 has, he will always choose L_2 so that all sequential equilibria yield payoff (3,1). It is also not difficult to prove that every perfect equilibrium is sequential, but again the converse does not hold. Consider Fig. 1a with $\alpha = \beta = 0$. In this game (BR,r) is a sequential equilibrium that is not perfect. (Note that (R,r) is an equilibrium of the subgame). Namely, if player 1 mistakenly chooses A and L with a positive probability, then l is strictly better than r so that perfectness requires 2 to choose l, but, consequently, only (AL,l) is a perfect equilibrium. It is easily checked that, in this example, the difference occurs only for a nullset of payoffs and Kreps and Wilson have shown that this holds generally. Formally, we have.

Proposition 3

(Selten (1976), Kreps and Wilson (1982a)).

(i) Every game has a perfect equilibrium.

(ii) Every perfect equilibrium is sequential and every sequential equilibrium is subgame perfect.

(iii) Generically, almost all sequential equilibria are perfect and the set of sequential equilibrium outcomes coincides with the set of perfect equilibrium outcomes.

Unfortunately, since the concept of sequential equilibria virtually does not restrict the beliefs at unreached information sets, it allows incredible beliefs which in turn may sustain 'unreasonable' equilibria. To illustrate this claim, consider Fig. 1a with $\alpha < 0$, $\beta > 0$, such that (AL,l) is the unique subgame perfect equilibrium. Now, change the game such that the 2 decision points of player 1 are condensed into 1 so that 1 chooses between AL, AR and B simultaneously and such that player 2 is not reached when 1 chooses B. If player 1 is a single player and not a team, one can argue that this modification does not change the strategic situation, however, it does change the set of sequential equilibria. In the modified game, (B,l) is a sequential (even perfect) equilibrium as player 2 can justify playing l by believing that 1 has chosen AR in case he is unexpectedly reached. However, these beliefs are nonsensical : if $\alpha < 0$, then AR is dominated by both AL and B so that 2 believes that 1 has chosen his worst action ! If 2 is reached, it only makes sense to believe that 1 has chosen AL and (AL,l) is the unique sensible equilibrium. Note that this *Forwards Induction* argument is based on a dominance relationship that is present only in the normal form, hence, concepts as sequential and perfect equilibria, which are based on the agent normal form, fail to incorporate it. This suggests to use the normal form to construct formal criteria which 'reasonable' beliefs should satisfy. In the sections 5 and 6 we will investigate whether this approach is feasible, however, let us remark that in the literature also much effort has been devoted to constructing intuitive, ad hoc, criteria to eliminate unreasonable sequential equilibria for specific classes of games. For a survey of the latter, we refer to Cho and Kreps (1987) and Van Damme (1987, Ch.10).

To conclude this section, we will show that the main results obtained depend crucially on the restriction to finite games. Let Γ be the (infinite) game in which first chance chooses $a \in \{0,2\}$ (both possibilities with probability 1/2), next player 1 (knowing a) chooses $x \in [0,2]$ and finally player 2 (knowing both a and x) chooses $y \in [0,2]$. The payoffs are

$$H_1^a(x,y) = (x-a)(y-a) \qquad H_2^a(x,y) = (1-x)y.$$

Note that player 2's payoff does not depend on a and that, in a subgame perfect equilibrium, this player chooses $y = 2$ (resp. $y = 0$) for $x < 1$ (resp. $x > 1$). Consequently, if $a = 0$, player 1 can guarantee himself slightly less than 2 by choosing x slightly less than 1 and it follows that player 1 does not have a best response unless player 2 chooses $y = 2$ if $a = 0$ and $x = 1$. Hence, the subgame $a = 0$ has a unique subgame perfect equilibrium in which player 1 chooses $x = 1$ and 2 responds with $y = 2$. We see that, in infinite games, not every equilibrium of a subgame can be extended to an overall equilibrium : if $a = 0$ and $x = 1$ then $y = 0$ is also optimal for player 2, but this action is not part of an equilibrium of the game with $a = 0$. By a similar reasoning one sees that the subgame $a = 2$ has a unique equilibrium in which 1 chooses $x = 1$ and 2 responds with $y = 0$. Hence, to keep 1 in equilibrium, player 2 has to base his decision at $x = 1$ upon whether $a = 0$ or $a = 2$ even though this information is fully irrelevant for player 2's own decision problem. Consequently, this game does not have a subgame consistent equilibrium and Proposition 2 (i) is not valid for infinite games. (Hellwig and Leininger (1987) have shown that perfect information games with compact action spaces always have a subgame perfect equilibrium as long as payoffs are continuous). Next, change the game so that player 2 is not informed about the outcome of the chance move. Then, in a sequential equilibrium (there are no proper subgames now) player 2 will still choose $y = 2$ (resp, $y = 0$) for $x < 1$ (resp. $x > 1$), however, as this player cannot condition on a, player 1 does not have a best response in at least one of the cases so that no sequential equilibrium exists, hence Proposition 3 (i) is not valid for infinite games. It should be remarked that Nash equilibria do exist in the modified game. Namely, if $\varepsilon \leqq 1/3$, then the strategy pair

$$x = \begin{cases} 1-\varepsilon & \text{if } a = 0 \\ 1+\varepsilon & \text{if } a = 2 \end{cases} \qquad y(x) = \begin{cases} 2 & \text{if } x \leqq 1-\varepsilon \\ 0 & \text{if } x \geqq 1+\varepsilon \\ (1+\varepsilon-x)/\varepsilon & \text{otherwise} \end{cases}$$

is a Nash equilibrium. Note that player 2's response is suboptimal in the intervals $(1-\varepsilon,1)$ and $(1,1+\varepsilon)$ but that player 1 chooses to avoid these intervals. (For general existence theorems for Nash equilibria in infinite games, we refer to Tijs (1981).

5. EQUILIBRIA IN NORMAL FORM GAMES

As we are mainly interested in extensive games, attention will be confined in this section to those normal form concepts that have also proved relevant for games in extensive form. In this section, we restrict ourselves to definitions and normal form properties; the connections to the extensive form will be discussed in section 6.

Perhaps the most natural and intuitive idea in game theory is iterative elimination of dominated (and/or payoff equivalent) strategies and to regard as unreasonable those equilibria that do not survive such a reduction of the game. Formally, a strategy σ_i' is said to be *dominated* by σ_i'' if $H_i(\sigma\backslash\sigma_i') \leqq H_i(\sigma\backslash\sigma_i'')$ for all σ with at least 1 inequality being strict. Unfortunately, the outcome of the process may depend on the order of elimination and there need not exist an equilibrium that survives all possible elimination orders. This may be illustrated by the 3-person game in which player i (i = 1,2) has the pure strategies L_i, M_i and R_i, in which player 3 chooses between L_3 and R_3 and in which the payoff functions are given by

$$H_1(L_1,L_3) = 2 \qquad H_1(R_1) = 1,$$
$$H_2(R_2,R_3) = 2 \qquad H_2(L_2) = 1,$$
$$H_3(M_1,L_2,L_3) = H_3(M_1,R_2,L_3) = H_3(L_1,M_2,R_3) = H_3(R_1,M_2,R_3) = 1,$$
$$H_i(\sigma) = 0 \quad \text{otherwise.}$$

(When we write $H_1(L_1,L_3) = 2$, this means that the payoff is independent of 2's strategy). The elimination order M_1,L_3,L_2,M_2,L_1 leaves only (R_1,R_2,R_3) with payoffs (2,1,0), while the order M_2,R_3,R_1,M_1,R_2 leads to (L_1,L_2,L_3) with payoffs (1,2,0), hence different orders yield different unique outcomes. Note, however, that player 3's payoff is 0 in both outcomes and it is not difficult to see that the 2 outcomes lie in the same connected component of equilibria (cf. Proposition 1). Proposition 6 shows that this is not a coincidence as there exists a component that remains for all possible elimination orders (i.e. for all elimination orders there always remains an equilibrium from this component).

Next, let us turn to perfect equilibria. Again the idea is that there are small trembles in the equilibrium strategies so that each player should choose a best response against slight perturbations of the equilibrium. (It should be remarked that such perturbations arise naturally if one accepts Harsanyi's argument (section 3) that players are always somewhat uncertain about their opponents' payoffs (see Van Damme (1983, Ch.5))). Formally, a mixed strategy equilibrium σ is said to be a *perfect equilibrium* if there exists a sequence $\{\sigma^k\}_k$ of completely mixed (i.e. interior, each pure strategy occurs with positive probability) strategy combinations such that σ is a best reply against σ^k for all k. From the definition it is clear that a perfect equilibrium strategy is undominated; the converse only holds in the 2-person case as shown in Van Damme (1983). To prove existence of perfect equilibria, an alternative characterization is useful. For $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_n)$ with $\varepsilon_i > 0$ (all i) and τ a completely mixed strategy combination, define the *perturbed game* $\Gamma(\varepsilon,\tau)$ as the game in which each player i when he intends to play σ_i actually plays $(1-\varepsilon_i)\sigma_i + \varepsilon_i\tau_i$. Selten (1976) has shown that σ is a perfect equilibrium if and only if σ is a limit point of equilibria of a sequence $\Gamma(\varepsilon^k,\tau^k)$ of perturbed games as ε^k tends to zero. Since each perturbed game is an ordinary normal form game with payoffs close to those of Γ, since the Nash correspondence is upper hemi-continuous and since the set of mixed strategies is compact, it follows that any normal form game has a perfect equilibrium. Hence,

<u>Proposition 4</u>

(Selten (1976), Van Damme (1983)).

Every normal form game has a perfect equilibrium. Perfect equilibria are undominated and in 2-person normal form games, every undominated equilibrium is perfect.

A perfect equilibrium need not survive iterated elimination of dominated strategies as the game of Fig. 1b with $\alpha < 0$ and $\beta > 0$ demonstrates.

AR is strictly dominated and in the reduced game r is weakly dominated so that only (AL,l) remains. However, also (BL,r) is perfect in Fig. 1b : if player 2 believes that the mistake AR is more likely than the mistake AL, then indeed it is optimal to play r. (Also cf. the discussion on plausible beliefs in section 4; to verify perfectness of (BL,r) one may also use Proposition 5 : the equilibrium is undominated for $\alpha \leqq 6$ and $\beta > 0$). Note that this example also shows that a perfect equilibrium need not remain perfect after strictly dominated strategies have been eliminated.

Myerson (1978) has argued that the above beliefs about mistakes are unreasonable. His argument is that, since AR is dominated by AL (we continue to assume $\alpha < 0$, $\beta > 0$), the mistake AR is more costly so that players will try much harder to prevent this mistake and that, therefore, it should be considered much less likely that this mistake actually occurs. (Van Damme (1983, Ch.5) has shown that this assumption can only be partially justified by means of the payoff uncertainty argument of section 3). The concept of proper equilibria formalizes this idea. For $\varepsilon > 0$, an *interior* strategy combination σ is said to be an ε-proper equilibrium if it satisfies

$$\text{if } H_i(\sigma\backslash s_i) < H_i(\sigma\backslash s_i'), \text{ then } \sigma_i(s_i) \leqq \varepsilon\, \sigma_i(s_i')$$

for all players i and all *pure* strategies s_i, s_i'. In words: if s_i yields less than s_i', then the mistake s_i occurs with a probability that is at most ε times the probability of the mistake s_i'. A strategy combination σ is a *proper equilibrium* if it is a limit point of ε-proper equilibria as ε tends to zero. Myerson (1978) has shown that every normal form game has a proper equilibrium and clearly every such equilibrium is perfect. Unfortunately, also proper equilibria need not survive when dominated strategies are eliminated. Take Fig. 1b with $0 < \alpha < 2$ and $\beta > 0$. Then iterated elimination still yields (AL,l), but now also (BL,r) is a proper equilibrium. Namely, if $\alpha > 0$, then AR is a better response against r than AL is, so that, when player 2 intends to play r, properness leads this player to believe that AR is more likely, but then indeed the choice r is justified. This example also shows that a proper equilibrium need not remain proper after a strictly dominated strategy (in this case AR) has been eliminated. Furthermore, a proper equilibrium need not remain proper when available mixed strategies are explicitly introduced as pure ones. Let M be the mixture 1/2 AL + 1/2 BL and add M as a pure strategy of player 1 in Fig. 1b. If $\alpha < 1$, then M dominates AR so that, according to properness, M should be considered much more likely. However, then player 2 should choose l as l yields more against M than r does, hence, in the modified game only (AL,l) is proper. Consequently, 2 normal form games with the same reduced normal form need not have the same sets of proper equilibria. That there need be no relation at all between properness and iterated elimination is once more demonstrated by Fig. 4a : Iterated elimination leads to (L_1,L_2) but this equilibrium is not proper. If 1 intends to play L_1, then R_2 is a better response than M_2, hence, player 1 should consider R_2 more likely, but then he should play R_1. (The unique proper equilibrium is $((3/4,1/4),L_2))$.

	L_2	M_2	R_2
L_1	2,2	1,0	0,1
R_1	2,2	0,3	1,0

Fig. 4a

	L_2	M_2	R_2
L_1	4,4	4,0	0,2
R_1	0,0	0,4	1,2

Fig. 4b

While in the previous discussion, we emphasized the drawbacks of the properness concept, it should be remarked that an important positive property will be discussed in section 6 (Proposition 8). Proposition 5 summarizes the above.

Proposition 5

(Myerson (1978), Kohlberg and Mertens (1986)).

Every normal form game has a proper equilibrium. Every proper equilibrium is perfect, but the converse does not hold. A proper equilibrium need not survive iterated elimination of dominated strategies and when elimination leads to a unique equilibrium, then this need not be proper, although it is perfect. Two games with the same reduced normal form need not have the same sets of proper equilibria.

The above discussion naturally suggests to try to remedy the drawbacks of properness by requiring a 'reasonable' equilibrium to be stable against all perturbations in strategies. Following Okada (1981) call an equilibrium σ *strictly perfect* if for any interior strategy combination τ the perturbed game $\Gamma(\varepsilon,\tau)$ has an equilibrium close to σ for ε close to zero. Unfortunately, self-enforcing equilibria need not be strictly perfect and strictly perfect equilibria need not exist. In Fig. 4a, if $\tau_2 \approx (0,1,0)$ then $\Gamma(\varepsilon,\tau)$ only has (L_1,L_2) as an equilibrium while the unique equilibrium is (R_1,L_2) if $\tau_2 \approx (0,0,1)$ so that no equilibrium is strictly perfect. Kohlberg and Mertens (1986) have suggested to apply the strict perfectness criterion to sets of equilibria rather than to singletons, and they call a set E a *stable set of equilibria* if it is a minimal set with the property that for each τ the game $\Gamma(\varepsilon,\tau)$ has an equilibrium close to E for sufficiently small ε. (Minimality is needed since otherwise the set of all equilibria would trivially satisfy the condition). Kohlberg and Mertens show that stable sets always exist, that they contain only perfect equilibria and that there exist stable sets that lie completely within one component (hence, it makes sense to speak of *stable components* and stable outcomes, cf. Proposition 1). In Fig. 4a, any perturbed game either has an equilibrium close to (L_1,L_2) or close to $((1/3,2/3),L_2)$ so that the unique stable set contains exactly these 2 points (i.e. the extreme points of the set of all equilibria). Consequently, a stable set need not contain a proper equilibrium. It is, however, conjectured that a stable component (i.e. a component that contains a stable set) always contains a proper equilibrium. Stable sets relate nicely to iterated elimination of dominated strategies as it can be shown that a stable set contains a stable set of any game obtained by elimination of a dominated strategy. In this statement, 'contains' cannot be replaced by 'is' as Fig. 4a demonstrates : R_2 is dominated and in the reduced game only $\{(L_1,L_2)\}$ is stable, hence, $((1/3,2/3),L_2)$ vanishes but this is needed in the original game for perturbed games $\Gamma(\varepsilon,\tau)$ with $\tau \approx (0,0,1)$. Furthermore, it can be shown that a stable set E contains a stable set of any game obtained by deletion of a pure strategy that is not a best reply against any element in E. We will return to this important property (which is called *Forward Induction* in Kohlberg and Mertens (1986) in section 6). A drawback of stability may be illustrated by means of Fig. 4b : Stable sets may vanish when inferior strategies are eliminated from the game. A strategy is said to be *inferior* if it is a best reply only against a strict subset of strategies where also some other strategy is a best reply, and Harsanyi (1978) has argued that such strategies should be eliminated. In Fig. 4b, the strategy R_2 is inferior (it is a best reply only against (1/2,1/2)) when R_2 is eliminated, R_1 is strictly dominated so that only (L_1,L_2) remains. In the original game, however, there are completely mixed equilibria in which player 2 chooses R_2 with probability 4/5 and player 1 randomizes (1/2,1/2) and any such equilibrium is strictly perfect, hence, is stable as a singleton set. (Note that this game has an even number of stable components).

The following Proposition summarizes the main properties of stable equilibria.

Proposition 6

(Kohlberg and Mertens (1986)).

Every normal form has a stable set of equilibria and even a stable component. A stable set contains only perfect equilibria. In generic normal form games, every equilibrium as a singleton is stable. A stable set contains a stable set of any game obtained by deletion of a strategy that is dominated or that is not a best reply against the set, however, stable sets may vanish if inferior strategies are eliminated.

To conclude this section, let us consider another set valued solution concept, that of persistent equilibria introduced in Kalai and Samet (1984). If Γ is a normal form game, then $C = (C_1,\dots,C_n)$ is said to be a retract if C_i is a nonempty, closed, convex set of strategies for each player i. A retract C is essential if there exists a neighborhood U of C (in the set of mixed strategy combinations) such that for any $\sigma \in U$ there exists a best reply against σ that is in C. A *persistent retract* is defined as a minimal essential retract and a *persistent equilibrium* is a Nash equilibriusm that belongs to a persistent retract. To illustrate this concept, consider Fig. 5. In Fig. 5a (Battle of the Sexes) the 2 strict equilibria

(L_1,L_2) and (R_1,R_2) are persistent. This game also admits a completely mixed equilibrium but the only essential retract that contains this equilibrium is the set of all mixed strategy combinations and this is not minimal (it contains $\{(L_1,L_2)\}$) so that the mixed equilibrium is not persistent. Hence, contrary to stability, requiring persistency may eliminate completely mixed equilibria. In Fig. 5b (matching pennies) there are no 'corner' equilibria that 'threaten' the completely mixed equilibrium and in this case the set of all strategies is a persistent retract, so that the mixed equilibrium is persistent. Fig. 5a already illustrates that the concept of persistency may not be appropriate for single shot games which are played without any possibilities for coordination as in this case only the mixed equilibrium is focal. Persistent equilibria can perhaps be best interpreted as stationary states when the game is played repeatedly but each time with different players who only have statistical information about the past. However, also this interpretation meets with difficulties as there exist examples (available from the author upon request) in which evolutionarily stable strategies (Maynard Smith (1982)) fail to be persistent.

	L_2	R_2
L_1	3,1	0,0
R_1	0,0	1,3

Fig. 5a

	L_2	R_2
L_1	4,-4	-4,4
R_1	-4,4	4,-4

Fig. 5b

It can be shown that persistent equilibria exist, but they need not be perfect. The latter may be shown by replacing (0,0) in Fig. 3 with the matching pennies game from Fig. 5b and constructing the associated normal form. In this game, only the set of all strategies is essential, so that all equilibria (including the imperfect one in which 2 threatens to play matching pennies) are persistent. Kalai and Samet, however, have also shown that there always exists an equilibrium that is both proper and persistent and by adapting their arguments one can even show that there exists a persistent retract that contains a stable set. Finally, Theorem

4 in Kalai and Samet (1984) implies that a persistent retract contains a persistent retract of any game obtained by deletion of a dominated strategy. The following Proposition summarizes these properties.

Proposition 7

(Kalai and Samet (1984)).

Every normal form has a persistent retract that contains a proper equilibrium as well as a persistent retract that contains a stable set. However, a persistent equilibrium need not be perfect, nor need a stable set contain a persistent equilibrium. A persistent retract contains a persistent retract of any game obtained by deletion of a dominated strategy.

6. EXTENSIVE FORM OR NORMAL FORM ?

In section 4 it has been shown that the extensive form concepts of sequential and perfect equilibria are unsatisfactory because they fail to incorporate dominance relationships that are present in the normal form. Now, the reason that these concepts were defined by means of the agent normal form is that Kreps/Wilson and Selten had the impression that backwards induction cannot be captured in the normal form. In this section, we will first show that this impression is not completely correct, although indeed some information is lost in the normalisation process. Next, it will be investigated whether a 'sensible' extensive form solution concept can be based on the normal form.

In section 4 we have seen that generic extensive games with perfect information have a unique sequential equilibrium. Since the roll back procedure from the extensive form can be mimiced in the normal form (first eliminate only those strategies that prescribe a suboptimal choice at terminal decision points, etc.), there exists an elimination order that reduces the normal form to its unique sequential equilibrium payoff. Propositions 1 and 6 guarantee that this payoff always remains and it can be shown that full reduction always (i.e. no matter the elimination order used) leaves just this payoff. Hence, it might seem that, at least for these simple games, knowledge of the normal form suffices. However, iterated elimination is only guaranteed to yield the correct payoff, by picking the wrong order one may retain strategies that are not even equivalent to the unique sequential equilibrium (although they are in the same component by Proposition 1). For example, in Fig. 6, the unique sequential equilibrium is (L_1r, R_2), but if $\alpha < 2$, then elimination starting with R_1r leaves only L_2 for player 2 and L_2 is not equivalent to R_2. Furthermore, if a game is not fully generic, iterated elimination may even produce the wrong outcome : If $\alpha = 2$ in Fig. 6, one may retain (R_1l, L_2) and this yields the wrong payoff. Hence, it seems that, even for perfect information games, extensive form analysis might be preferable.

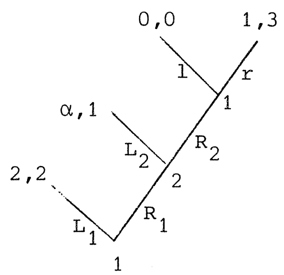

Fig. 6

	L_2	R_2
L_1l	2,2	2,2
L_1r	2,2	2,2
R_1l	α,1	0,0
R_1r	α,1	1,3

This example already shows that one cannot expect normal form analysis to yield the exact sequential equilibrium strategies : In the normal form one does not have to care about own mistakes, so that L_1l and L_1r appear as duplications, even though only L_1r is sequentially rational. Hence, the best one can hope for is to obtain normal form strategies that are equivalent to a sequential equilibrium. Fig. 6 also shows that such strategies are not obtained by requiring normal form perfectness : If $\alpha < 2$, then (L_1l, L_2) is perfect in the normal form, but L_2 is not equivalent to R_2. In this example, all normal form perfect equilibria still yield the sequential equilibrium outcome (2,2), but it is easy to construct examples in which even this property fails. Let $(\alpha,\beta) = (-1,1)$ in Fig. 1. Then (AL,l) is the unique subgame perfect equilibrium of the extensive form, but (BL,r) is an undominated, hence, perfect equilibrium of the normal form and this yields an entirely different outcome. Note that to sustain (BL,r) in the normal form, player 2 must consider the mistake AR to be more likely than the mistake AL and these beliefs are clearly nonsensical from the extensive game point of view (as L dominates R at player 1's second information set), yet such beliefs are not excluded by normal form perfectness. These beliefs are excluded by the properness concept and indeed, for $\alpha < 0$, the unique sequential equilibrium of Fig. 1a is the unique proper equilibrium of Fig. 1b. Furthermore, in Fig. 6, properness requires player 2 to choose R_2 so that also in this example each proper equilibrium is equivalent to the sequential equilibrium. More generally, it can be shown that the restrictions that normal form properness imposes on mistakes imply sequentially rational behavior at all information sets in the extensive form except at those that a player prevents by his own actions. Formally

Proposition 8

(Kohlberg and Mertens (1986), Van Damme (1984)).

If σ is a proper equilibrium of the normal form, then σ is equivalent to a sequential equilibrium of the extensive form so that, in particular, $\mathbb{P}(\sigma)$ is a sequential equilibrium outcome.

Note that, as Proposition 8 is confined to equivalence classes of strategies, the 'normal form' can be replaced by the 'semi reduced normal form' in this statement. However, the result no longer holds for proper equilibria of the reduced normal form (cf. Proposition 5, also see Kohlberg and Mertens (1986)). Furthermore, for nongeneric games, 'sequential' cannot be replaced by perfect (Van Damme (1984)). In general, not all sequential equilibria correspond to normal form proper equilibria. Consider the modification of Fig. 1a in which player 1 chooses between AL, AR and B simultaneously, player 2 not being reached if 1 chooses B, and let $(\alpha,\beta) = (-1,1)$. This does not change the normal form so that again only (AL,l) is proper, however, (B,l) is perfect in the extensive form. Hence, Proposition 8 points to the possibility of using properness as a criterion for eliminating 'unreasonable' sequential equilibria. Clearly, this criterion is allowed only when the normal form is indeed appropriate, i.e. when each player is a single individual (cf. section 2) and this excludes games with incomplete information (Van Damme (1987,Ch.10) shows that for such games, properness involves unjustified comparisons of utility between different types of the same player). However, as properness fails to incorporate the forward induction logic (see Proposition 5), this criterion is bound to be unsuccessfull. Again consider Fig. 1 but now let $(\alpha,\beta) = (1,1)$. Then iterated elimination reduces the normal form to (AL,l) and we argued that only this equilibrium is sensible, yet also (BL,r) is proper in the normal form.

Several comments are in order concerning Forwards Induction :

(i) Applying forwards induction is justified only if normal form analysis is indeed appropriate, i.e. if there is a central player coordinating the agents. For example, in Fig. 1 with $\alpha = 1$ and $\beta > 0$, if the agents of player 1 are completely independent, then the agent normal form is relevant and the equilibrium (BR,r) is self-enforcing since a deviation of the first agent of player 1 does not necessarily trigger a deviation of the second agent. This again brings us to the issue of whether the normal form representation assumes self-commitment. Suppose $\alpha = 1$ and β is large so that (α,β) risk dominates (3,1) (Harsanyi and Selten (1987)) implying that, where only the subgame to be played, the equilibrium (R,r) would most likely result. In this case, if player 1 actually reaches his second decision point, will he continue with the plan AL or will he reoptimize (hence, act as a seperate agent) and decide to choose R after all ? (Note that as soon as, the second decision point is reached, B is no longer available and R is not dominated). Hence, it is not completely clear that forward induction is justified not even in the case where each player is a single individual.

(ii) Forward Induction is not completely captured by iterated elimination of dominated strategies, since, by replacing a terminal node by a subgame with a unique equilibrium having this value, one can destroy all dominance relationships. For example, if one replaces (0,0) in Fig. 3 by matching pennies (Fig. 5b), then the normal form does not have dominated pure strategies. Applying this same trick also shows that forward induction is not captured by the combination of persistency and properness : In Fig. 1a, let $(\alpha,\beta) = (1,1)$ and replace the endpoint after (AR,l) by matching pennies. Then forwards induction still yields (AL,l) as the solution (if 2 is reached, he should conclude 1 has chosen AL since AR, followed by optimal play in matching pennies, is dominated by B), but in the normal form, the unique retract is the set of all strategies, so that the outcome (2,4) is both persistent and proper. It seems that the formal criterion of iterated elimination of non best replies best captures the intuitive idea of forward induction (see Van Damme (1987, Ch.10) for a justification of this criterion).

(iii) Adding a little bit of incomplete information may completely destroy the forward induction argument (just as it does with backwards induction). Consider Fig. 1a, but assume that (α,β) is either (1,1) with probability $1-\varepsilon$ or (4,1) with small, but positive probability ε and that only player 1 knows which case prevails. In the modified game, player 2 is justified in choosing r for he might think that $\alpha = 4$ and that player 1 has chosen AR, hence, the forward induction logic does not apply. Specifically, in the modified game a strict equilibrium results when player 1 chooses B (resp. AR) if $\alpha = 1$ (resp. $\alpha = 4$) and player 2 chooses r (strict means that each player looses by deviating), hence, with very high probability we obtain the outcome (2,4) that was not viable in the complete information case (see Fudenberg, Kreps and Levine (1987) for an extension of this argument).

(iv) Forward induction is incompatible with subgame consistency (see Abreu and Pearce (1984)). Let Γ_i be the game in which player i ($i = 1,2$) chooses between the 'easy way out' with payoff (2,2) or playing the battle of the sexes game of Fig. 5a; if player j ($j \neq i$) has to move, he knows that i has not taken the easy way out. Consequently, when j moves, he should conclude that i will play his favorite equilibrium in Fig. 5a (otherwise i would have taken the easy way out), so that forward induction yields (3,1) (resp. (1,3)) as the outcome of Γ_1 (resp, Γ_2). Now, let Γ be the game in which first a coin is tossed to determine which player has the easy way out, both players being informed about the chance move. Iterated elimination of dominated strategies in the normal form of Γ yields a unique solution : The player who has the easy way out does not use this option but rather he plays battle of the sexes and receives his most preferred equilibrium. Clearly, this equilibrium is not subgame consistent (as this condition requires that the same equilibrium played in Γ_1 and Γ_2).

The above discussion leaves us only with stability as a normal form criterion to select 'reasonable' sequential equilibria in the extensive form. Indeed, stability seems to capture forwards induction (Proposition 6), but, unfortunately, a stable set of the normal form need not contain a sequential equilibrium of the extensive game. Fig. 4a can be viewed as the normal form of the extensive game in which player 2 first chooses between L_2 or to play the subgame with strategies L_1,M_1,M_2 and R_2. The unique subgame perfect equilibrium in this extensive game requires player 1 to choose (3/4,1/4) (this is the proper equilibrium of the normal form) and we have already seen that this does not belong to the stable set. In this example, the stable set yields the correct outcome (2,2) but one can construct examples (see Kohlberg and Mertens (1986)) in which even this property fails. However, such examples involve stable sets that are not contained in one component and one can show (J.F. Mertens, private communication) that stable components of normal forms associated with generic extensive games always contain a proper equilibrium. Combining this result with the Propositions 1, 6 and 8 we see that, at least for generic extensive games, stability might constitute a criterion to eliminate 'unreasonable' sequential equilibrium outcomes. We write 'might' as it has not yet been completely cleared whether the forward induction argument is acceptable and whether stability indeed eliminates all unreasonable outcomes (cf. Fig. 4b). Furthermore, it should be remarked that stability only deals with outcomes not with strategies in the extensive form, and this is the price we pay for using a normal form concept : We cannot tell what the player should do off the equilibrium path (unless we select a proper equilibrium from the stable component). Finally, it should be remarked that, for nongeneric games, stable sets might be too large. Let $\alpha = 2$ in Fig. 6 and replace (0,0) by a decision of player 2 between (0,0) and (3,-1). Then, when applying backwards induction, one never has indifference to the payoff (2,2),however,it is easily seen that the stable set contains the outcome (R_1,L_2) and this yields the wrong payoff.

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CHAPTER II

EQUILIBRIUM POINTS OF BIMATRIX GAMES

by Mathijs Jansen

1. INTRODUCTION

The theory of two-person non-cooperative games is concerned with the behavior of two individuals (players) dealing with a conflict situation where binding agreements are impossible. One of the purposes of the theory is to prescribe a unique solution to the problem of how these individuals should behave in such a situation. Since binding agreements are impossible, one property of a solution to a non-cooperative game is that neither player has the incentive to unilaterally deviate from that solution. Therefore the study of two-person non-cooperative games is based on the equilibrium point concept (Nash (1950)). An equilibrium point of a game is a pair of strategies, one for each player, whereby no player can gain by deviating. Unfortunately, the equilibrium point concept has several disadvantages, because in general equilibria are not unique, interchangeable, or Pareto optimal. So the problem arisis which equilibrium point should be chosen as the solution of a game when it has more than one equilibrium point. In the literature, we come across several ways to handle this problem.

1. One shrinks (refines) the equilibrium point set by imposing an extra condition.
2. One tries to find a (numerical) procedure for selecting a particular equilibrium point which can serve as the solution of the game.
3. One tries to formulate a list of attractive properties so that each game has *exactly one* equilibrium point satisfying all the properties.

In this paper we are concerned with the first area of research. Furthermore we will restrict our attention to bimatrix games, a type of non-cooperative game where the two players have a finite number of pure strategies. Section 2 begins with the definition of a bimatrix game and an equilibrium point. The two examples in this section form the introduction to section 3 in which we investigate, for a bimatrix game, the structure of the set of equilibria. Section 4, 5, 6 and 7 will deal with several refinements of the equilibrium point concept, restricted to the two-person case.

This paper only includes proofs in the case of a new result or proof technique.

Notation. The elements of the basis of unit vectors of $\mathbb{R}^m$ are denoted by $e_1,\dots,e_m$. For a finite set S, $|S|$ is the number of elements of S. The convex hull of a set $S \subset \mathbb{R}^m$ is denoted by conv(S). If $C \subset \mathbb{R}^m$ is a convex set, then we write ext(C), dim(C) and relint(C) for the set of extreme points of C, the dimension of (the affine hull of) C and the relative interior of C, respectively. Finally,

$$S^m := \{p \in \mathbb{R}^m;\ p \geq 0 \text{ and } \Sigma_{i=1}^m p_i = 1\}.$$

2. BIMATRIX GAMES AND EQUILIBRIUM POINTS

We start with the formal definition of a bimatrix game.

Let A and B two m×n-matrices. The two-person game where the players 1 and 2 choose, independently of each other, a $p \in S^m$ and a $q \in S^n$, respectively and subsequently receive the payoff

$$pAq = \Sigma_{i=1}^m \Sigma_{j=1}^n p_i a_{ij} q_j$$

and

$$pBq = \Sigma_{i=1}^m \Sigma_{j=1}^n p_i b_{ij} q_j, \quad \text{respectively}$$

is called the m×n-*bimatrix game* corresponding to the ordered pair of matrices. This game is denoted by (A,B).

With this game we can model a conflict situation in which two individuals are involved. The first individual can choose one of m alternatives numbered 1,2,..,m, while the second one can choose one of n alternatives numbered 1,2,..,n. If player 1 chooses the *pure* strategy $i \in \{1,2,\dots,m\}$ and player 2 chooses the *pure* strategy $j \in \{1,2,\dots n\}$, they receive the reward a_{ij} and b_{ij}, respectively. For a vector $p \in S^m$ ($q \in S^n$) p_i (q_j) can be seen as the probability of choosing i (j) and pAq (pBq) can be interpreted as the expected reward for player 1 (2) corresponding to the *(mixed) strategies* p and q.

As described before, the study of two-person noncooperative games is based on pairs of strategies from which no player can gain by deviating unilaterally. Hence we focus our attention to equilibrium points, where a pair $(\bar{p},\bar{q}) \in S^m \times S^n$ of strategies is called an *equilibrium point* of the $m \times n$-bimatrix game (A,B) if $\bar{p}$ is a *best reply* to $\bar{q}$:

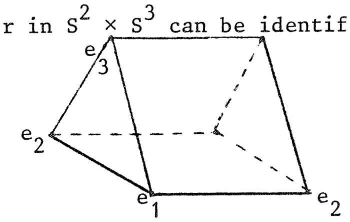
$$\bar{p}A\bar{q} = \max_p pA\bar{q}$$

and $\bar{q}$ is a *best reply* to $\bar{p}$:

$$\bar{p}B\bar{q} = \max_q \bar{p}Bq.$$

The set of equilibrium points of the bimatrix game (A,B) is denoted by E(A,B).
So $(\bar{p},\bar{q}) \in E(A,B)$ if and only if $(\bar{p},\bar{q}) \in B_1(\bar{q}) \times B_2(\bar{p})$, where $B_1(\bar{q})$ $(B_2(\bar{p}))$ is the set of all best replies to $\bar{q}$ $(\bar{p})$, or, in other words, if and only if $(\bar{p},\bar{q})$ is a *fixed point* of the mapping that assigns to a pair $(p,q) \in S^m \times S^n$ the set $B_1(q) \times B_2(p)$. By applying the "fixed point theorem" of Kakutani to this mapping, Nash (1950) proved that for all bimatrix games the set of equilibria is nonempty.

In order to get an impression of the shape of the set of equilibria, we make a picture of this set for two examples of 2×3-bimatrix games. To that purpose the strategy set S^2 is identified with a line segment. Each point of this line segment corresponds in an obvious way with a strategy of player 1. For instance, each endpoint of the segment corresponds with one of the strategies $e_1 = (1,0)$ or $e_2 = (0,1)$. Analogously, the strategy set S^3 is identified with an equilateral triangle. In this way any strategy pair in $S^2 \times S^3$ can be identified with a point in the following diagram

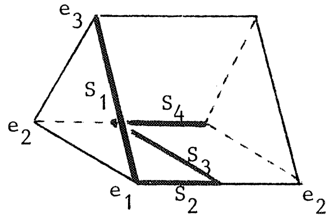


Example 2.1

For the 2×3-bimatrix game (A,B), where

$$A := \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \qquad \text{and} \qquad B := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

the equilibrium point set is connected and composed of four convex pieces S_1, S_2, S_3 and S_4 :

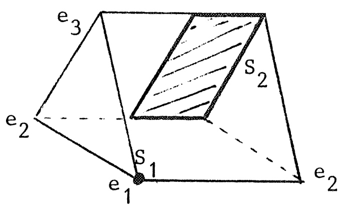


Example 2.2

For the 2×3-bimatrix game (A,B), where

$$A := \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix} \qquad \text{and} \qquad B := \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$$

the equilibrium point set consists of two disjunct convex pieces S_1 and S_2 :



The sets defined in the following equivalence play an important role in section 3 of this paper.

Lemma 2.1 A strategy pair (p,q) is an equilibrium point of the bimatrix game (A,B) if and only if

$$C(p) \subset M(A;q) \text{ and } C(q) \subset M(p;B),$$

where $C(p) := \{i;\ p_i > 0\}$ is the *carrier* of p, C(q) is the carrier of q, $M(A;q) := \{i;\ e_i Aq = \max_k e_k Aq\}$ is the set of *pure best replies* to q and M(p;B) is the set of pure best replies to p.

Finally, we note that for a bimatrix game (A,B) and strategies p of player 1 and q of player 2

$B_1(q) = \text{conv}\{e_i;\ i \in M(A;q)\}$

and

$B_2(p) = \text{conv}\{e_j;\ j \in M(p;B)\}$.

3. THE STRUCTURE OF THE EQUILIBRIUM POINT SET

A bimatrix game modelling a situation where the interests of the players are strictly opposed is of the form (A,-A). Such a game is usually called a *matrix game* and denoted by A. In 1928, Von Neumann proved in his famous minimax theorem that each matrix game A has a *value*

$$\text{val}(A) := \max_p \min_q pAq = \min_q \max_p pAq$$

and that the *optimal strategy spaces* of player 1

$O_1(A) := \{p;\ pAq \geqq \text{val}(A), \text{ for all } q\}$

and of player 2

$O_2(A) := \{q;\ pAq \leqq \text{val}(A), \text{ for all } p\}$

are nonempty.

It is well known that $E(A,-A) = O_1(A) \times O_2(A)$ and that $O_i(A)$ is a convex polytope for $i \in \{1,2\}$.

The elements of the set $\text{ext}(O_i(A))$, called *extreme optimal strategies* of player $i \in \{1,2\}$, were characterized by Shapley and Snow (1950).

Bohnenblust, Karlin and Shapley (1950) and Gale and Sherman (1950) established a relationship between the dimensions of the sets of optimal strategies.

Furthermore the same authors completely answered the question which pairs of convex polytopes can serve as the sets of optimal strategies for some matrix game.

Several authors such as Vorob'ev (1958), Kuhn (1961), Millham (1972, 1974), Winkels (1979) and Jansen (1981a) generalized (some of) these results to the class of bimatrix games.

In this section we have collected most of the results of these writers concerning the structure of the set of equilibria of a bimatrix game.

3.1 Maximal Nash subsets

Some important concepts are gathered in the following

Definition 3.2

Two equilibrium points (p,q) and (p',q') of a bimatrix game (A,B) are *interchangeable* (cf. Nash (1951)) if (p,q') and (p',q) are also equilibria of (A,B). A set of equilibria is a *maximal Nash subset* for the game (A,B) if it is a maximal set with the property that all its elements are interchangeable.

Note that the maximal Nash subsets for the bimatrix game introduced in example 2.1 are S_1, S_2, S_3 and S_4. For the bimatrix game introduced in example 2.2 these sets are S_1 and S_2.

The term maximal Nash subset was introduced by Heuer and Millham (1976). Nash, who already considered such sets in 1951, called them *subsolutions*. Winkels (1979) used the term *Nash component*. These authors showed that a maximal Nash subset for an $m \times n$ - bimatrix game is a closed and convex subset of $S^m \times S^n$. In Jansen (1981a) and Winkels (1979) it appeared that a maximal Nash subset is in fact the Cartesian product of two convex polytopes :

Lemma 3.1 Let S be a maximal Nash subset for the $m \times n$ - bimatrix game (A,B). If $(\mathring{p}, \mathring{q}) \in \text{relint}(S)$, then

$$S = K(\mathring{q}) \times L(\mathring{p}),$$

where $K(\mathring{q}) := \{p \in S^m;\ (p,\mathring{q}) \in E(A,B)\}$ and $L(\mathring{p}) := \{q \in S^n;\ (\mathring{p},q) \in E(A,B)\}$ are convex polytopes.

The examples in section 2 suggest that the equilibrium point set of a bimatrix game is the finite union of connected components or even the finite union of convex components, where

Definition 3.2 A connected (convex) component of the equilibrium point set is a maximal connected (convex) subset of the set of equilibria.

In Jansen (1981a) it was shown that every pair of equilibrium points from a convex subset of the set of equilibria of a bimatrix game is interchangeable. As a consequence we have

Theorem 3.1 (Cf. Jansen (1981a)). Let C be a convex subset of the set of equilibrium points of a bimatrix game. Then C is a convex component if and only if C is a maximal Nash subset.

If (p,q) is an equilibrium point of a bimatrix game (A,B), {(p,q)} is a convex subset of E(A,B). Hence we can find, applying Zorn's lemma, a convex component of E(A,B) containing (p,q). Consequently, in view of theorem 3.1, every equilibrium point of the game (A,B) is contained in a maximal Nash subset and E(A,B) is the union of such subsets.
Before we show that the number of maximal Nash subsets for a bimatrix game is finite, we need the following lemma.

Lemma 3.2 (Cf. Jansen (1981a)). Let S be a maximal Nash subset for the bimatrix game (A,B) and let $(\mathring{p},\mathring{q}) \in \text{relint}(S)$. Then $(p,q) \in S$ if and only if $C(p) \subset C(\mathring{p})$, $C(q) \subset C(\mathring{q})$, $M(A;q) \supset M(A;\mathring{q})$ and $M(p;B) \supset M(\mathring{p};B)$.

This result implies that a maximal Nash subset S with $(\mathring{p},\mathring{q}) \in \text{relint}(S)$ is completely determined by the quartet $(C(\mathring{p}),\ M(A;\mathring{q}),\ C(\mathring{q}),\ M(\mathring{p};B))$. Since there is only a finite number of such quartets, we have a proof of

Theorem 3.2 (Cf. Jansen (1981a), Winkels (1979)). The set of equilibrium points of a bimatrix game is a (not necessarily disjunct) union of a finite number of maximal Nash subsets.

As we see in the first example of section 2, for two different maximal Nash subsets S_1 and S_2 the set $S_1 \cap S_2$ may be non-empty. Heuer and Millham (1979) proved that in that case $S_1 \cap S_2$ is a (proper) face of the convex polytopes S and T.

As a consequence of theorem 3.2 each connected component C of the equilibrium point set of a bimatrix game (A,B) is of the form

$$C = S_1 \cup S_2 \cup \ldots \cup S_n,$$

where $S_1, S_2, \dots, S_n$ are maximal Nash subsets for (A,B). So every bimatrix game has a finite number of connected components. This result was also obtained by Kohlberg and Mertens (1986). In their paper the same authors introduced the bimatrix game (A,B), where

$$A := \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & -1 \\ 1 & 0 & -2 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -2 \end{bmatrix} .$$

For this game $E(A,B) = S_1 \cup S_2 \cup \dots \cup S_6$ is a connected component for which $S_i \cap S_{i+1} \neq \emptyset$ for $i = 1,2,\dots,5$ and $S_1 \cap S_6 \neq \emptyset$.

In 1974, Millham derived, for a special case, dimension relations for maximal Nash subsets for bimatrix games. His results were extended by Jansen (1981a). In the description of the dimension of (the affine hull of) a maximal Nash subset S for a bimatrix game (A,B) an important role is played by the matrices

$$A(S) := [a_{ij}]_{i \in M(A;q), j \in C(q)} \quad \text{and} \quad B(S) := [b_{ij}]_{i \in C(p), j \in M(p;B)}$$

where $(p,q) \in \text{relint}(S)$. By lemma 3.2 these so-called S-*submatrices* of A and B, respectively do not depend on the choice of the point $(p,q) \in \text{relint}(S)$.

Theorem 3.3 Let S be a maximal Nash subset for the bimatrix game (A,B) with $A > 0$ and $B > 0$ and let $(p,q) \in \text{relint}(S)$. Then

$$\text{dimension } L(p) = | C(q) | - \text{rank } A(S)$$

and

$$\text{dimension } K(q) = | C(p) | - \text{rank } B(S).$$

3.2 Extreme equilibria

In view of theorem 3.2, the set of equilibrium points of a bimatrix game can be found if the extreme points of the maximal Nash subsets are known. These so-called *extreme equilibrium points* can be characterized by means of certain square submatrices of the payoff matrices :

Theorem 3.4 (Cf. Vorob'ev (1958), Kuhn (1961), Jansen (1981a)).
If (p,q) is an extreme equilibrium point of the $m \times n$-bimatrix game (A,B) and $\gamma := |C(q)|$, then there exists a $\gamma \times \gamma$-submatrix K of A such that (renumber, if necessary, the rows and columns of A in such a way that K is in the upper left corner of A)

(1) the $(\gamma+1) \times (\gamma+1)$-matrix $\tilde{K} := \begin{bmatrix} K & 1 \\ -1 & 0 \end{bmatrix}$ is nonsingular,

(2) $q_j = (\det \tilde{K})^{-1} \Sigma_{i=1}^{\gamma} K_{ij}$ if $j \in C(q)$ and (K_{ij} is the cofactor of the element k_{ij}),

(3) $pAq = \det K / \det \tilde{K}$.

An analogous statement can be formulated w.r.t. the connection of the vector p and the number pBq with a certain square submatrix of B.

The extreme equilibrium points of an $m \times n$-bimatrix game (A,B) can also be found by calculating the extreme points of two polyhedral sets associated with that game. In order to introduce these sets, we note that Mills (1960) and Mangasarian and Stone (1964) proved that a pair (p,q) of strategies is an equilibrium point of the game (A,B) if and only if there exist scalars α and β such that

$$e_i Aq \leq \alpha, \quad \text{for all } i \in \mathbb{N}_m$$

$$pBe_j \leq \beta, \quad \text{for all } j \in \mathbb{N}_n$$

and

$$p(A + B)q = \alpha + \beta.$$

This result led to the definition of the convex polyhedral sets :

$$P_B := \{(p,\beta) \in S^m \times \mathbb{R};\ pBe_j \leq \beta, \text{ for all } j \in \mathbb{N}_n\}$$

$$Q_A := \{(q,\alpha) \in S^n \times \mathbb{R};\ e_i Aq \leq \alpha, \text{ for all } i \in \mathbb{N}_m\}.$$

The same authors observed that all the equilibrium points of a bimatrix game (A,B) can be found by solving the *quadratic programming problem*

$$\underset{(p,\beta,q,\alpha)}{\text{maximize}}\ p(A + B)q - \alpha - \beta$$

$$\text{such that } (p,\beta,q,\alpha) \in P_B \times Q_A.$$

In Jansen (1981a) a proof can be found of the following

Lemma 3.3 Let (A,B) be a bimatrix game and let $(p,\beta,q,\alpha) \in P_B \times Q_A$.

Then (p,q) is an extreme equilibrium point of the game (A,B) if and only if

$(p,\beta) \in \text{ext } P_B$,
$(q,\alpha) \in \text{ext } Q_A$,
$\alpha = pAq$ and $\beta = pBq$.

Based on these results, Winkels (1979) has developped a method to find the whole set of equilibrium points of a bimatrix game.
Firstly the extreme points of the polyhedral sets P_B and Q_A associated with the game (A,B) must be determined, for example with the help of the algorithm of Balinsky.
Then the extreme points $(p^1,\beta^1),(p^2,\beta^2),\ldots$ of P_B and $(q^1,\alpha^1),(q^2,\alpha^2),\ldots$ of Q_A are conveniently arranged in the following tableau :

<table>
<tr><td></td><td></td><td>$q^1 \ldots q^j \ldots$</td></tr>
<tr><td></td><td></td><td>$\alpha^1 \ldots \alpha^j \ldots$</td></tr>
<tr><td>p^1
.
p^i
.
.</td><td>β^1
.
β^i
.
.</td><td>.
.
.
.. k_{ij} ...
.</td></tr>
</table>

The left (upper) block of this tableau contains as rows (columns) the extreme points of P_B (Q_A). If one defines

$$k_{ij} = \begin{cases} 1 & \text{if } \alpha^j = p^iAq^j \text{ and } \beta^i = p^iBq^j \\ 0 & \text{otherwise,} \end{cases}$$

then the matrix K in the lower right block contains all the necessary information about the extreme equilibrium points, because for an element $(p^i,\beta^i,q^j,\alpha^j) \in P_B \times Q_A$, $k_{ij} = 1$ if and only if (p^i,q^j) is an extreme equilibrium point.
Finally, Winkels describes an efficient method how on the basis of the matrix K all maximal Nash subsets for the game can be constructed.

Consider the bimatrix game (A,B) as described in the second example of section 2. For this game the associated polyhedral sets are

$$Q_A = \{(q,\alpha) \in S^3 \times \mathbb{R};\ q_1 + 2q_2 + 2q_3 \leqq \alpha\}$$

and

$$P_B = \{(p,\beta) \in S^2 \times \mathbb{R};\ 2p_1 \leqq \beta,\ 2p_2 \leqq \beta\}.$$

Since

$$\text{ext}(Q_A) = \{(e_1,1),(e_2,2),(e_3,2)\}$$

and

$$\text{ext}(P_B) = \{(e_1,2),(e_2,2),((\tfrac{1}{2},\tfrac{1}{2}),1)\}$$

the corresponding tableau is

			1	0	0
			0	1	0
			0	0	1
			1	2	2
1	0	2	1	0	0
$\frac{1}{2}$	$\frac{1}{2}$	1	0	1	1
0	1	2	0	1	1

From this tableau we conclude that the extreme equilibrium points are (e_1,e_1), $(e_2,e_3),(e_2,e_2),((\frac{1}{2},\frac{1}{2}),e_2)$ and $((\frac{1}{2},\frac{1}{2}),e_3)$,
while

$$E(A,B) = S_1 \cup S_2$$

with

$$S_1 = \{(e_1,e_1)\}$$

$$S_2 = \text{conv}\{(\tfrac{1}{2},\tfrac{1}{2}),e_2\} \times \text{conv}\{e_2,e_3\}.$$

4. REFINEMENTS OF THE EQUILIBRIUM POINT CONCEPT

In the first section we have given several reasons to refine the equilibrium point concept as introduced by Nash. The last 15 years several refinements of this concept have been proposed in the literature. In the following we will distinguish three kinds of refinements. These will be investigated successively in the sections 5, 6 and 7.

(1) In section 5, we will deal with equilibrium points that are stable (in a sense to be defined later on) against small perturbations of

the strategy spaces of the game. In this field an important role is played by (strictly) perfect equilibria.

(2) In section 6, the best reply sets are central. We are interested in equilibrium points for which the best reply sets satisfy some stability condition(s). We deal with (quasi-)strong, regular, robust and persistent equilibria.

(3) Finally, in section 7, we pay attention to equilibria that are stable against slight perturbations of the payoff(matrice)s of the game. We come accross essential and (strongly) stable equilibria.

5. (STRICTLY) PERFECT EQUILIBRIA

In 1975, Selten assumes that each player with a small probability makes a mistake, that is whenever he chooses some (pure) strategy in fact some close by completely mixed strategy is played. Therefore Selten is interested in equilibrium points for which each player's equilibrium strategy is not only a best reply against the equilibrium strategy of his opponent, but also against *some* slight perturbation of this strategy. In order to investigate such "perfect equilibria", Selten models the idea of making mistakes via a perturbed game, i.e. a game in which the players' strategies are restricted to simplices with faces parallel to the faces of the original simplex.

Definition 5.1 For an m×n-bimatrix game (A,B) and a *mistake vector* $\varepsilon \in \mathbb{R}^{m\times n}$ (that is : $\varepsilon > 0$, $\Sigma_{i=1}^{m} \varepsilon_i < 1$ and $\Sigma_{j=1}^{n} \varepsilon_{m+j} < 1$), the ε-*perturbed game* is the game where each strategy p [q] of player 1 [2] is replaced by the convex combination

$$p(\varepsilon) := (1 - \Sigma_{i=1}^{m} \varepsilon_i)p + (\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m)$$

$$[q(\varepsilon) := (1- \Sigma_{j=1}^{n} \varepsilon_{m+j})q + (\varepsilon_{m+1},\ldots,\varepsilon_{m+n})]$$

of the strategy p [q] and the completely mixed strategy

$$\sigma := \frac{1}{\Sigma_{i=1}^{m} \varepsilon_i} (\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m) \ [\tau := \frac{1}{\Sigma_{j=1}^{n} \varepsilon_{m+j}} (\varepsilon_{m+1},\ldots,\varepsilon_{m+n})].$$

This results in a payoff $p(\varepsilon)Aq(\varepsilon)$ for player 1 and $p(\varepsilon)Bq(\varepsilon)$ for player 2.

In the ε-perturbation of an $m\times n$-bimatrix game (A,B) both players only choose completely mixed strategies, since the probability of choosing row i (column j) is at least ε_i (ε_{m+j}). Hence the strategy space of player 1 is

$$\{p \in S^m;\ p_i \geqq \varepsilon_i \text{ for all } i \in \mathbb{N}_m\},$$

whereas

$$\{q \in S^n;\ q_i \geqq \varepsilon_{m+j} \text{ for all } j \in \mathbb{N}_n\}$$

is the strategy space of player 2.

Definition 5.2 Let (A,B) be an $m\times n$-bimatrix game. An equilibrium point (p,q) is called *perfect* if there exists a sequence $\{\varepsilon(k)\}_{k \in \mathbb{N}}$ of mistake vectors converging to 0 and a sequence $\{(p(k),q(k))\}_{k \in \mathbb{N}}$ of elements of $S^m \times S^n$ converging to (p,q) such that, for each $k \in \mathbb{N}$, $(p(k),q(k))$ is an equilibrium point of the $\varepsilon(k)$-disturbed game.

Selten (1975) obtained the following characterization of perfect equilibria.

Theorem 5.1 An equilibrium point (p,q) of an $m\times n$-bimatrix game (A,B) is perfect if and only if there exists a sequence $\{(p(k),q(k))\}_{k \in \mathbb{N}}$ in $S^m \times S^n$ converging to (p,q) such that, for all $k \in \mathbb{N}$,

(1) $p(k)$ and $q(k)$ are completely mixed

(2) p is a best reply against $q(k)$ and q is a best reply against $p(k)$.

In order to prove that every bimatrix game has at least one perfect equilibrium point, we show that there corresponds with the ε-perturbed game of an $m\times n$-bimatrix game (A,B) an $m\times n$-bimatrix game $(A(\varepsilon),B(\varepsilon))$ such that $(\bar{p},\bar{q}) \in E(A(\varepsilon),B(\varepsilon))$ if and only if $(\bar{p}(\varepsilon),\bar{q}(\varepsilon))$ (as defined in definition 5.1) is an equilibrium point of the ε-perturbed game (Cf. Van Damme (1983), theorem 2.4.3).

Take

$$A(\varepsilon) = [a_{ij}(\varepsilon)]_{i=1,\ j=1}^{m\quad n},$$

where for $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$

$$a_{ij}(\varepsilon) := e_i(\varepsilon)Ae_j(\varepsilon).$$

Then, if we use the notation $\lambda := \Sigma_{i=1}^{m}\ \varepsilon_i$ and $\mu := \Sigma_{j=1}^{n}\ \varepsilon_{m+j}$,

$$\begin{aligned} p(\varepsilon)Aq(\varepsilon) &= [(1-\lambda)p + \lambda\sigma]A[(1-\mu)q + \mu\tau] \\ &= (1-\lambda)(1-\mu)pAq + \lambda(1-\mu)\sigma Aq + \mu(1-\lambda)pA\tau + \lambda\mu\sigma A\tau \\ &= \Sigma_{i,j}p_i[(1-\lambda)(1-\mu)e_iAe_j + \lambda(1-\mu)\sigma Ae_j + \mu(1-\lambda)e_iA\tau + \lambda\mu\sigma A\tau]q_j \\ &= \Sigma_{i,j}p_i[e_i(\varepsilon)Ae_j(\varepsilon)]q_j = \Sigma_{i,j}p_ia_{ij}(\varepsilon)q_j = pA(\varepsilon)q. \end{aligned}$$

This implies that

$$\bar{p}(\varepsilon)A\bar{q}(\varepsilon) \geqq p(\varepsilon)A\bar{q}(\varepsilon), \quad \text{for all } p \in S^m$$

if and only if

$$\bar{p}A(\varepsilon)\bar{q} \geqq pA(\varepsilon)\bar{q}, \quad \text{for all } p \in S^m.$$

A similar result can be derived for the matrix $B(\varepsilon)$ defined analogously. This completes our proof.

Since the ε-perturbed game of a bimatrix game (A,B) is equivalent with a bimatrix game (close to (A,B)), each ε-perturbed game possesses at least one equilibrium point. In view of the compactness of the strategy spaces, each sequence of equilibrium points as mentioned in definition 5.2 has a limit point. Since such a limit point is an equilibrium point, we have a proof of

Theorem 5.2 (Selten (1975)). Every bimatrix game has at least one perfect equilibrium point.

With the help of theorem 5.1 it is easy to see that a perfect equilibrium point is undominated, where

Definition 5.3 An equilibrium point $(\bar{p},\bar{q})$ of an m×n-bimatrix game (A,B) is *undominated* if for all $p \in S^m$ and $q \in S^n$

$$pA \geqq \bar{p}A \quad \text{implies} \quad pA = \bar{p}A$$

and

$$Bq \geqq B\bar{q} \quad \text{implies} \quad Bq = B\bar{q}.$$

By using the theory of matrix games, Van Damme (1983) established the following characterization.

Theorem 5.3 An equilibrium point of a bimatrix game is perfect if and only if it is undominated.

Tijs (1985) obtained the same result by using geometrical arguments. He further shows that the set of perfect equilibria of a bimatrix game is a (not necessarily disjunct) union of a finite number of convex polytopes.

For both games in the examples 2.1 and 2.2 the set of perfect equilibria coincides with the maximal Nash subset S_1. This is an easy consequence of the fact that for both games e_1 is the only undominated strategy for player 1.

Unfortunately the perfectness concept does not eliminate all unreasonable equilibria. Myerson (1978) showed that adding strictly dominated strategies may enlarge the set of perfect equilibria. For that reason he introduced so-called *proper equilibria*. We will not discuss this equilibrium point concept in this paper. Furthermore, by restricting to perfect equilibrium points, one may eliminate equilibria with attractive payoffs. For the game (A,B), where

$$A := \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$$

(e_1,e_1) is a perfect equilibrium point yielding both players a lower payoff than the imperfect equilibrium point (e_2,e_2).

For these reasons Okada (1981) proposed to study equilibria which are not only stable against *some* mistakes made by a player, but against *all* mistakes. Formally :

Definition 5.4 An equilibrium point (p,q) of the m×n-bimatrix game (A,B) is called *strictly perfect* if for any mistake vector ε there exists an equilibrium point (p(ε),q(ε)) of the ε-disturbed game corresponding to (A,B) such that $\lim_{\varepsilon \downarrow 0} (p(\varepsilon),q(\varepsilon)) = (p,q)$.

The bimatrix game (A,B) (Van Damme (1983)), where

$$A := \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B := \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

shows that there exist games without strictly perfect equilibria. Kohlberg and Mertens (1986) however, proved that all bimatrix games possess at least one *strictly perfect set* of equilibria, where

Definition 5.5 A closed set C of equilibria of an m×n-bimatrix game (A,B) is strictly perfect if for any open set V containing C there exists a neighbourhood U of $0 \in \mathbb{R}^{m \times n}$ such that, for all mistake vectors ε ∈ U, the ε-disturbed game corresponding to (A,B) has an equilibrium point in V.

One easily shows that a strictly perfect set contains a perfect equilibrium point. Furthermore, if C is a strictly perfect set and (p,q) ∈ C is imperfact, then in view of the closedness of the set of perfect equilibria, there exists a neighbourhood W of (p,q) such that
(1) all equilibrium points in W are imperfect
(2) C\W is a strictly perfect set.
This implies that a strictly perfect set not properly contained in another one, contains only perfect equilibria. Kohlberg and Mertens (1986) called such minimal strictly perfect sets *stable sets* and showed that every bimatrix game has at least one stable set and that such sets possess several attractive properties.

6. BEST REPLY STABLE EQUILIBRIA

In this section we deal with equilibria which are not only a fixed point of the best-reply-multifunction $(p,q) \to B_1(q) \times B_2(p)$ mentioned in section 2 but with equilibria for which the best-reply-multifunction satisfies some extra condition. Successively, we pay attention to (quasi-)strong equilibria, regular equilibria, robust equilibria and persistent equilibria.

6.1. Strong and quasi-strong equilibria

In 1973, Harsanyi introduced equilibria for which every player's equilibrium strategy is the *only* best reply to the strategy of his opponent.

Definition 6.1 An equilibrium point (p,q) of a bimatrix game (A,B) is called *strong* if $|B_1(q)| = |B_2(p)| = 1$.

Since $B_1(q) = \text{conv}\{e_i;\ i \in M(A;q)\}$ and $B_2(p) = \text{conv}\{e_j;\ j \in M(p;B)\}$, lemma 2.1 implies that an equilibrium point (p,q) is strong if and only if

(1) p and q are pure strategies (that is : $|C(p)| = |C(q)| = 1$)

(2) (p,q) is a quasi-strong equilibrium point,

where

Definition 6.2 An equilibrium (p,q) of a bimatrix game (A,B) is called *quasi-strong* if

$$C(p) = M(A;q) \text{ and } C(q) = M(p;B).$$

Such equilibria with the property that no player has a pure best reply other than the pure strategies belonging to the carrier of his equilibrium strategy, were also introduced by Harsanyi in his papers of 1973. The author (1981b,c) further investigated such equilibria and paid special attention to equilibrium points that are quasi-strong and isolated, where an equilibrium point (p,q) of a bimatrix game (A,B) is called *isolated* if there exists a neighbourhood V of (p,q) such that

$E(A,B) \cap V = \{(p,q)\}$
or, equivalently, if $\{(p,q)\}$ is a maximal Nash subset for that game. With the help of theorem 3.3 one obtains the following characterization.

Lemma 6.1 Let (p,q) be a quasi-strong equilibrium point of a bimatrix game (A,B) with A > 0 and B > 0. Then (p,q) is isolated if and only if $|C(p)| = |C(q)|$ and the matrices $[a_{ij}]_{i \in C(p),\ j \in C(q)}$ and $[b_{ij}]_{i \in C(p),\ j \in C(q)}$ are nonsingular.

From the foregoing characterization of strong equilibria it will be clear that the set of strong equilibria of a game may be empty. It is still an open problem if all bimatrix games possess a quasi-strong equilibrium point. For a bimatrix game with a quasi-strong equilibrium point the following result was obtained (cf. Jansen (1981b)).

Theorem 6.1 If (p,q) is a quasi-strong equilibrium point of a bimatrix game (A,B) and (p,q) ∈ S, where S is a maximal Nash subset for (A,B), then (p,q) ∈ relint(S).

In view of theorem 3.2 this implies that a *quasi-strong bimatrix game*, that is a game for which *all* equilibria are quasi-strong, has a finite number of equilibrium points. As a consequence of the following result of Meister (1984) a quasi-strong bimatrix game has in fact an odd number of equilibrium points.

Theorem 6.2 A bimatrix game with a finite number of equilibrium points has an odd number of quasi-strong ones.

This theorem also implies that a game with a finite number of equilibria possesses a quasi-strong equilibrium point (Cf. corollary 7.9 in Jansen (1981b)). An other result concerning the existence of quasi-strong equilibria was obtained by Jansen (1981b).

Theorem 6.3 A bimatrix game with a convex set of equilibria (a Nash solvable game) has at least one quasi-strong equilibrium point.

For a bimatrix game with (p,q) as its unique equilibrium point theorem 6.3 and lemma 6.1 imply that

(1) (p,q) is a quasi-strong equilibrium point

(2) $|C(p)| = |C(q)|$.

So we have a proof of one part of the following result of Kreps (1974).

Theorem 6.4 Let $(p,q) \in S^m \times S^n$ be given. Then there exists an m×n-bimatrix game (A,B) with (p,q) as its unique equilibrium point if and only if $|C(p)| = |C(q)|$.

To finish this section, we consider a type of bimatrix game for which quasi-strongness appears in a natural way.

If an m×n-bimatrix game (A,B) is *completely mixed*, that is $C(p) = \mathbb{N}_m$ and $C(q) = \mathbb{N}_n$, for all $(p,q) \in E(A,B)$, then obviously all equilibria are quasi-strong. Hence, it follows from theorem 6.1 that a completely mixed bimatrix game has only a finite number of equilibrium points. It is also easy to see that, in the completely mixed case, the set of equilibrium points is convex. Using these two facts the first statement of the following result of Raghavan (1970) and Heuer (1975) has been proved.

Theorem 6.5 Let (A,B) be a completely mixed bimatrix game. Then

(1) (A,B) has a unique equilibrium point,

(2) the matrices A and B are square,

(3) if $A > 0$ and $B > 0$, the matrices A and B are nonsingular.

6.2. Regular equilibria

In the literature several authors (Cf. Harsanyi (1973b), Van Damme (1983), Jansen (1987)) have introduced a regularity concept for equilibrium points. We will start with the definition given by the author. After that this definition will be compared with those given by Harsanyi and Van Damme.

As a first step we will associate with an m×n-bimatrix game a smooth map $f : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ for which there exists a one-to-one correspondence between the equilibrium points of that game and the so-called feasible solutions of the equation $f(z) = 0$. In order to define this map we will first show how the problem of finding an equilibrium point of an $m\times n$-bimatrix game (A,B) can be formulated as the (*linear complementarity*) *problem* LCP(A,B) : find, for the matrix $M := \begin{bmatrix} 0 & -A \\ -B^t & 0 \end{bmatrix}$, a vector $z = (x,y) \in \mathbb{R}^m \times \mathbb{R}^n$ such that

(1) $z \geqq 0$

(2) $w := Mz + 1_{m+n} \geqq 0$

(3) $\langle w,z \rangle = 0$ (where $\langle w,z \rangle$ denotes the inner product of w and z).

A vector z satisfying (1) and (2) is called *feasible* and a feasible vector satisfying also (3) is called a *solution* of the linear complementarity problem LCP(A,B) (Cf. Lüthi (1976)).

In the following lemma it is shown that there is a one-to-one correspondence between the equilibria of the game (A,B) and the nonzero solutions of LCP(A,B).

Lemma 6.2 Let (A,B) be an $m\times n$-bimatrix game with $A > 0$ and $B > 0$.

(a) If $(p,q) \in E(A,B)$, then $(p/pBq,\ q/pAq)$ is a solution of LCP(A,B).

(b) If $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$ is a nonzero solution of LCP(A,B), then $x \neq 0$, $y \neq 0$ and $(x/\Sigma_{i=1}^m x_i,\ y/\Sigma_{j=1}^n y_j) \in E(A,B)$.

Hence, in order to find the equilibria of the $m\times n$-bimatrix game (A,B), we have to find all vectors $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ satisfying

$$\begin{cases} e_iAy \leqq 1 \text{ and } x_i \geqq 0 & \text{for all } i \in \mathbb{N}_m \\ xBe_j \leqq 1 \text{ and } y_j \geqq 0 & \text{for all } j \in \mathbb{N}_n \qquad (\textit{feasibility}) \\ \Sigma_{i=1}^m x_i(1-e_iAy) + \Sigma_{j=1}^n y_j(1-xBe_j) = 0 & (\textit{complementarity}). \end{cases}$$

So if we define the map $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{m+n}$ by

$$f_k(x,y) := \begin{cases} x_k(1-e_kAy) & \text{if } 1 \leqq k \leqq m \\ y_{k-m}(1-xBe_{k-m}) & \text{if } m+1 \leqq k \leqq m+n \end{cases}$$

our problem is to find feasible vectors x and y with $\Sigma_{k=1}^{m+n} f_k(x,y) = 0$ or,

equivalently, $f(x,y) = 0$.
So we have reduced the problem of finding the equilibrium points of an m×n-bimatrix game to the determination of the nonzero feasible solutions of the equation $f(x,y) = 0$, where $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{m+n}$ is an infinitely often differentiable mapping associated with the game in question.
Let

$$J_f(x,y) = [\partial_j f_i(x,y)]_{i=1,\ j=1}^{m+n\ \ m+n}$$

the Jacobian of f evaluated at (x,y). Since one can expect that an equilibrium point will have nice properties if f is locally invertible at the solution (x,y) corresponding to that equilibrium point, i.e. if $J_f(x,y)$ is nonsingular, we introduced (Jansen (1987)) :

Definition 6.3 An equilibrium point (p,q) of a bimatrix game (A,B) is called *regular* if the Jacobian $J_f(x,y)$ is nonsingular, where (x,y) is the solution of the equation $f(x,y) = 0$ corresponding to (p,q) and where f is the mapping associated with (A,B).

Also Harsanyi (1973b), in order to introduce regular equilibria, associates with a given game a certain smooth mapping $g : \mathbb{R}^k \to \mathbb{R}^k$ ($k \in \mathbb{N}$ depending on the game) and considers nonnegative solutions of the equation $g(z) = 0$. Van Damme (1983) slightly modified Harsanyi's definition.
The approach of Harsanyi and Van Damme however has as a drawback that not every nonnegative solution of the equation $g(z) = 0$ is an equilibrium point and not every equilibrium point leads to a solution of the equation.
In view of Corollary 3.4.2 of Van Damme (1983) and the next theorem, our concept of regularity is equivalent to the one introduced by Harsanyi and Van Damme.

Theorem 6.5 (Jansen (1987)). An equilibrium point of a bimatrix game is regular if and only if it is isolated and quasi-strong.

By applying the Implicit Function Theorem to the map
$F : \mathbb{R}^{2mn} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{m+n}$ defined by

$$F_k(A,B,x,y) = \begin{cases} x_k(1-e_kAy) & \text{if } 1 \leq k \leq m \\ y_{k-m}(1-xBe_{k-m}) & \text{if } m+1 \leq k \leq m+n \end{cases}$$

one can show that in the neighbourhood of a point (A^*,B^*,p^*,q^*), where (p^*,q^*) is a regular equilibrium point of the bimatrix game (A^*,B^*), the set $\{(A,B,p,q);\ (p,q) \in E(A,B)\}$ is a smooth curve through the point (A^*,B^*,p^*,q^*).

Theorem 6.6 (Cf. theorem 2.5.5 of Van Damme (1983)). Let (p^*,q^*) be a regular equilibrium point of a bimatrix game (A^*,B^*). Then there exist neighbourhoods U of (A^*,B^*) and V of (p^*,q^*) such that

(1) $|E(A,B) \cap V| = 1$, for all $(A,B) \in U$

(2) the mapping $\sigma : U \to V$ defined by $\{\sigma(A,B)\} = E(A,B) \cap V$ is differentiable.

To finish this section we consider *regular* bimatrix games, i.e. games for which all equilibria are regular. In view of theorem 6.5 a bimatrix game is regular if and only if it is quasi-strong.
Furthermore, theorem 6.2 implies that a regular game has an odd number of equilibria (Cf. Harsanyi (1973b)). Since a non-degenerate (in the sense of Lemke and Howson (1964)) bimatrix game is regular and the class of non-degenerate m×n-bimatrix games is a dense subset of the set of all m×n-bimatrix games ,we have partially proved

Theorem 6.7 The set of regular m×n-bimatrix games is an open and dense subset of the set of all m×n-bimatrix games.

Proof. Let (A,B) be a regular m×n-bimatrix game and let $(p,q) \in E(A,B)$. Then, in view of theorem 6.6, there exist neighbourhoods U of (A,B) and V of (p,q) such that

$$|E(A',B') \cap V| = 1, \quad \text{for all } (A',B') \in U.$$

Since the number of equilibrium points of the game (A,B) is finite, one can choose the neighbourhood U of (A,B) and an $\varepsilon > 0$ in such a way that for any game $(A',B) \in U$

$$|E(A',B') \cap B_\varepsilon(p,q)| = 1, \quad \text{for all } (p,q) \in E(A,B).$$

In view of the upper semicontinuity of the multifunction E (assigning to an m×n-bimatrix game its set of equilibrium points),

$$E(A',B') \subset \bigcup_{(p,q) \in E(A,B)} B_\varepsilon(p,q), \quad \text{for all } (A',B') \in U.$$

So $|E(A,B)| = |E(A',B')|$, for all $(A',B') \in U$.
With the help of lemma 2.1 it can be shown that U can be choosen in such a way that all equilibrium points of any game (A',B') in U are quasi-strong. Hence, all games in the set U are regular which completes the proof. □

In fact we have shown in the proof of the foregoing theorem that the set

$$\{(A,B) \text{ is a regular game with } E(A,B) = k\}$$

is open for all $k \in \mathbb{N}$. Consequently the set of all regular m×n-bimatrix games is, as a finite union of open sets, disconnected and the number of equilibrium points is locally constant on the set of regular games.

Finally, we note that Harsanyi (1973) proved that almost all bimatrix games are regular by showing that the class of m×n-bimatrix games with an irregular equilibrium point has Lebesgue measure zero (where the class of all m×n-games is identified with $\mathbb{R}^{2mn}$).

6.3. Robust equilibria

From theorem 5.1 it appears that for a perfect equilibrium point (p,q) the equilibrium strategy p (q) is a best reply to some (completely mixed) strategies close to q (p). Based on the idea that for a reasonable equilibrium point the equilibrium strategy of a player remains a best reply if the other player changes his equilibrium strategy slightly but arbitrarily, Okada (1983) introduced robust equilibria. To be more precise

Definition 6.4 An equilibrium point $(\bar{p},\bar{q})$ of an m×n-bimatrix game (A,B) is *robust* if there exist neighbourhoods U of $\bar{p}$ and V of $\bar{q}$ such that

$$\bar{p} \in B_1(q), \qquad \text{for all } q \in V$$
$$\bar{q} \in B_2(p), \qquad \text{for all } p \in U.$$

Okada characterized the robust equilibria as follows.

Theorem 6.8 An equilibrium point $(\bar{p},\bar{q})$ of a bimatrix game (A,B) is robust if and only if the following conditions are fullfilled :

(1) $e_iA = e_kA$ if $i,k \in C(\bar{p})$,

(2) $Be_j = Be_k$ if $j,k \in C(\bar{q})$,

(3) if $i \in C(\bar{p})$ and $k \in M(A;\bar{q})\backslash C(\bar{p})$, then

$a_{ij} = a_{kj}$, for all $j \in C(\bar{q})$

$a_{ij} \geq a_{kj}$, for all $j \notin C(\bar{q})$,

(4) if $j \in C(\bar{q})$ and $k \in M(\bar{p};B)\backslash C(\bar{q})$, then

$b_{ij} = b_{ik}$, for all $i \in C(\bar{p})$

$b_{ij} \geq b_{ik}$, for all $i \notin C(\bar{p})$.

The theorem implies that a quasi-strong equilibrium point (p,q) is robust if and only if all pure strategies e_i with $i \in C(p)$ are equivalent for player 1 and if all pure strategies e_j with $j \in C(q)$ are equivalent for player 2. Furthermore a strong equilibrium is robust.

If for a robust equilibrium point (p,q) of a game (A,B) and mistake vector ε

$$p_i^{\varepsilon} := \begin{cases} p_i - \sum_{k \notin C(p)} \frac{\varepsilon_k}{|C(p)|} & \text{if } i \in C(p) \\ \varepsilon_i & \text{if } i \notin C(p) \end{cases}$$

and q^{ε} is defined in a similar way, then, for ε small enough, $(p^{\varepsilon},q^{\varepsilon})$ is an equilibrium point of the ε-perturbed game of (A,B). So we have proved the following

Theorem 6.9 A robust equilibrium point is strictly perfect.

Example 6.1

If $A := \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ and $B := \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, then (p,q) is a robust equilibrium point, where $p = q = (\frac{1}{2},\frac{1}{2},0)$.

6.4 Persistent equilibria

In 1984, Kalai and Samet generalized the robustness concept by strengthening the pointwise stability of equilibria to a notion of neighbourhood stability.

For an m×n-bimatrix game (A,B), they call a set P×Q, where $P \subset S^m$ and $Q \subset S^n$ are nonempty, closed, convex sets, an *essential Nash retract* for (A,B) if there exist neighbourhoods U of P and V of Q such that

(1) for any $q \in V$, a $\bar{p} \in P$ can be found with $\bar{p} \in B_1(q)$

(2) for any $p \in U$, a $\bar{q} \in Q$ can be found with $\bar{q} \in B_2(p)$.

With the help of Zorn's lemma, Kalai and Samet show that every bimatrix game has a *minimal* essential Nash retract. They are interested in equilibrium points contained in such retracts.

Definition 6.5 An equilibrium point is called *persistent* if it belongs to some minimal essential Nash retract.

Since for a robust equilibrium point (p,q), $\{p\} \times \{q\}$ is a minimal essential Nash retract, a robust equilibrium point is persistent. Moreover we have

Theorem 6.10 Every bimatrix game has a persistent equilibrium point.

Proof. Let P×Q be a minimal essential Nash retract and let $p^0 \in P$. By (2) we can find a $q^1 \in Q$ such that $q^1 \in B_2(p^0)$. In view of (1) there exists a $p^1 \in P$ with $p^1 \in B_1(q^1)$. By continuing this process, we can find sequences $p^0, p^1, \ldots$ in P and $q^1, q^2, \ldots$ in Q with for all $k \in \mathbb{N}$

$$q^k \in B_2(p^{k-1}) \text{ and } p^k \in B_1(q^k).$$

Since P and Q are compact, both sequences have a limit point, say p^* and q^*, respectively. Then $(p^*, q^*) \in P\times Q$ is an equilibrium point. □

Furthermore Kalai and Samet prove that every bimatrix game possesses a perfect and persistent equilibrium point.

7. ESSENTIAL AND STRONGLY STABLE EQUILIBRIA

In this section we deal with equilibrium points that are stable against slight perturbations of the payoffs.
Wu Wen-tsün and Jiang Jia-he (1962) called an equilibrium point of a bimatrix game essential if, roughly speaking, all games in a neighbourhood of the game in question have an equilibrium point close to it. To be more precise

Definition 7.1 An equilibrium point (p,q) of a bimatrix game (A,B) is *essential* if there exists, with every neighbourhood V of (p,q) a neighbourhood U of (A,B) such that

$E(A',B') \cap V \neq \emptyset$, for all $(A',B') \in U$.

Kojima, Okada and Shindoh (1985) strenghtened this stability concept by introducing equilibrium points that change continuously and uniquely against slight perturbations of the payoffmatrices of the players.

Definition 7.2 An equilibrium point (p,q) of a bimatrix game (A,B) is *strongly stable* if there exist neighbourhoods U of (A,B) and V of (p,q) such that

(1) $|E(A'B') \cap V| = 1$, for all $(A',B') \in U$, and

(2) the mapping $\sigma : U \to V$ defined by $\{\sigma(A',B')\} = E(A',B') \cap V$ is continuous.

Obviously, a strongly stable equilibrium point is isolated and essential.
That the concept of essentiallity is related with that of quasi-strongness follows from

Theorem 7.1 An essential equilibrium point of a bimatrix game is an element of some quasi-strong maximal Nash subset for that game (a maximal Nash subset is quasi-strong if all equilibrium points in the relative interior of the set are quasi-strong).

Proof. Let (p,q) be an essential equilibrium point of the bimatrix game (A,B). Now we can proceed as in the proof of the theorems 5.2 and 7.2 in Jansen (1981b) to construct a sequence $\tilde{p}^1, \tilde{p}^2, \ldots$ converging to p such that for k large enough

(1) $(\tilde{p}^k, q) \in E(A,B)$

(2) $C(\tilde{p}^k) = M(A;q)$.

Also a sequence $\tilde{q}^1, \tilde{q}^2, \ldots$ converging to q can be constructed in such a way that for k large enough

(3) $(p, \tilde{q}^k) \in E(A,B)$

(4) $C(\tilde{q}^k) = M(p;B)$.

Since the number of maximal Nash subsets for (A,B) is finite, we may suppose, without loss of generality, that there exists a maximal Nash subset S for (A,B) such that

$$(\tilde{p}^k, q), (p, \tilde{q}^k) \in S, \text{ for k large enough.}$$

Since S is closed, also $(p,q) \in S$. If $(\overset{\circ}{p}, \overset{\circ}{q}) \in \text{relint}(S)$, then in view of (2) and lemma 3.2, for k large enough,

$$M(A;\overset{\circ}{q}) \subset M(A;q) = C(\tilde{p}^k) \subset C(\overset{\circ}{p}) \subset M(A;\overset{\circ}{q}).$$

Hence $C(\overset{\circ}{p}) = M(A;\overset{\circ}{q})$ and similarly $C(\overset{\circ}{q}) = M(\overset{\circ}{p};B)$.

So S is a quasi-strong maximal Nash subset containing (p,q). □

As a consequence of this theorem, an isolated and essential equilibrium point is quasi-strong. Hence a strongly stable equilibrium point is regular. Since theorem 6.6 implies that a regular equilibrium point is strongly stable we have the following result (Cf. Jansen (1987)).

Theorem 7.2 An equilibrium point of a bimatrix game is strongly stable if and only if it is isolated and quasi-strong.

Because this result implies that an isolated and quasi-strong equilibrium point is essential, we have a new proof of

Theorem 7.3 (Cf. theorem 7.5, Jansen (1981b)). An isolated equilibrium point of a bimatrix game is essential if and only if it is quasi-strong.

The following result was also obtained by Jansen (1981b). For a different proof see theorem 3.4.5 of Van Damme (1983).

Theorem 7.4 An essential and quasi-strong equilibrium point of a bimatrix game is isolated.

Corollary 7.1 If (p,q) is an essential equilibrium point of a bimatrix game, then

(1) (p,q) is an isolated (and quasi-strong) equilibrium point, or

(2) (p,q) is an element of the relative boundary of some quasi-strong maximal Nash subset for that game.

As a consequence of this corollary the class of all essential $m \times n$-bimatrix games - a game is essential if all its equilibria are essential - coincides with the class of all regular $m \times n$-bimatrix games.

Since there are bimatrix games without an essential equilibrium point - take for A and B the $n \times n$-matrix ($n \in \mathbb{N} \backslash \{1\}$) with all coefficients equal to one - Jiang Jia-he (1964) investigated the essentiallity of the connected components of the equilibrium point set. He calls a closed set K of the equilibrium point set of a bimatrix game (A,B) essential if for any open set V containing K there exists a neighbourhood U of (A,B) such that

$$E(A',B') \cap V \neq \emptyset, \quad \text{for all } (A',B') \in U.$$

Obviously any closed set containing an essential equilibrium point is essential.

He obtained the following result also proved by Kohlberg and Mertens (1986).

Theorem 7.5 For every bimatrix game the equilibrium point set has at least one essential connected component.

Since for an essential component consisting of one point only, that point must be an essential equilibrium point, the following is immediate

Corollary 7.2 (Cf. Wu Wen-tsün, Jiang Jia-he (1962), Jansen (1981b), Meister (1984)). A bimatrix game with only a finite number of equilibrium points has at least one essential equilibrium point.

Suppose that K is an essential closed set of equilibria of the bimatrix game (A,B). Let for a mistake vector ε close to 0, $(A(\varepsilon),B(\varepsilon))$ be the bimatrix game equivalent with the ε-perturbed game. Since K is essential and since $(A(\varepsilon),B(\varepsilon))$ is close to (A,B), the game $(A(\varepsilon),B(\varepsilon))$ has an equilibrium point (p,q) close to K. So the equilibrium point $(p(\varepsilon),q(\varepsilon))$ of the ε-disturbed game induced by (p,q) (see definition 5.1) will also be close to K. This implies that an essential closed set is strictly perfect. Furthermore, our considerations at the end of section 5 imply

Theorem 7.6 Each essential connected component of the set of equilibria of a bimatrix game contains a stable set (and hence a perfect equilibrium point).

In a comparable way Van Damme (1983) proved the following result which implies that stability against perturbations in the payoff's implies stability against perturbations in strategies.

Theorem 7.7 An essential equilibrium point of a bimatrix game is strictly perfect.

Analogously to the proof of theorem 7.1 the following result can be obtained.

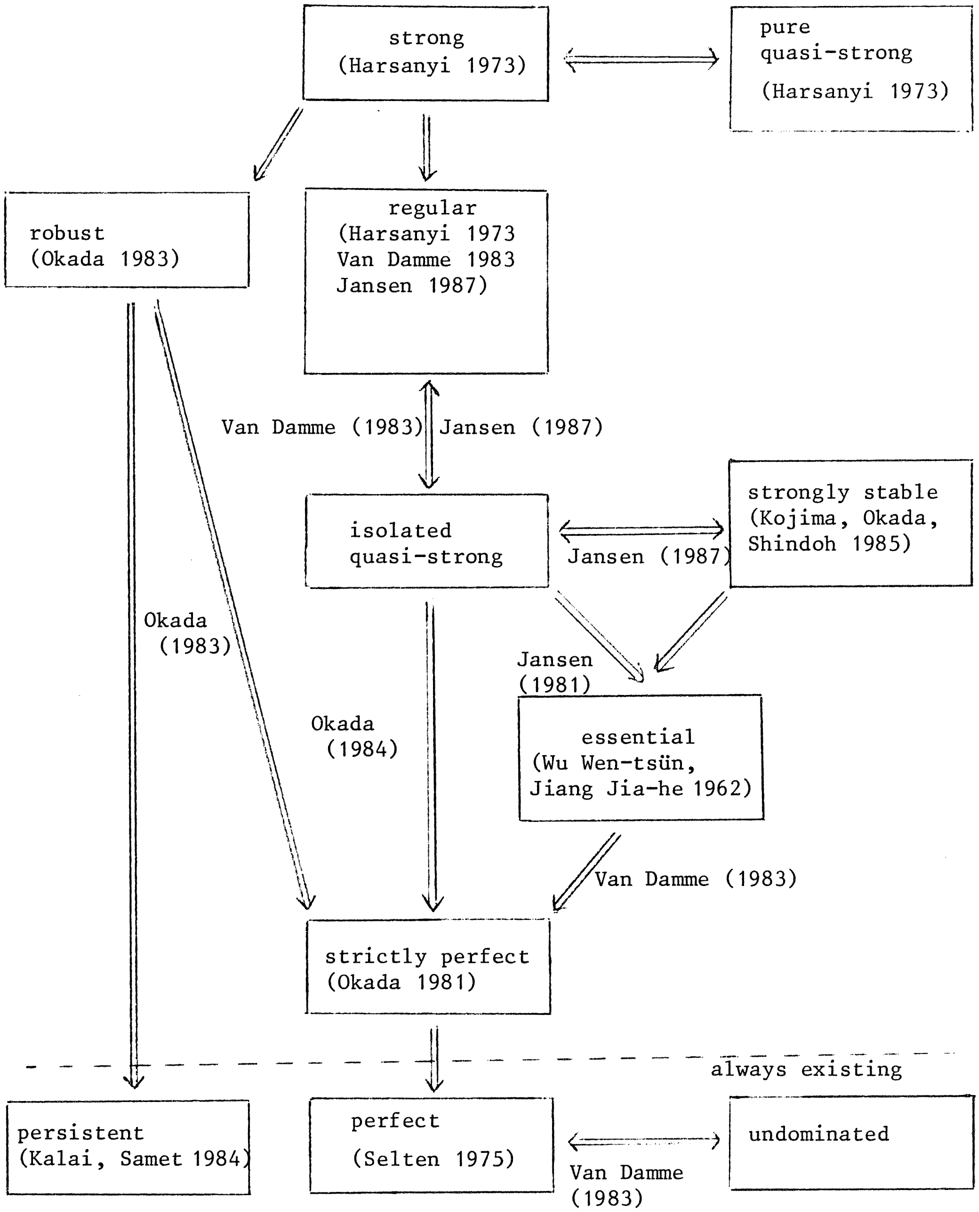
Theorem 7.8 An isolated and essential maximal Nash subset contains at least one quasi-strong equilibrium point.

In combination with theorem 7.5 this result implies

Corollary 7.3 A bimatrix game for which all maximal Nash subsets are pairwise disjunct has at least one quasi-strong equilibrium point.

This corollary generalizes theorem 6.3 and corollary 7.9 in Jansen (1981b).

Finally, we give for bimatrix games an overview of the relations between the refinements of the equilibrium concept mentioned in this paper.



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CHAPTER III

GAMES WITH INCOMPLETE INFORMATION

by Peter Borm

1. INTRODUCTION

In game theory a lot of attention is paid to games with complete information. These are games in which all participants (= players) have full information about the various actions the players can take, the actual "payoffs" (to all players) generated by a choice of action by each player, and the precise information available to the other players.

However, real life situations are different. Think about competing firms which often lack information about the other's financial means, capacity of production, costs of labour etc. Information also plays a crucial role in disarmament negotiations. Both sides don't know the exact quantity (how many) and quality (what types) of their opponent's armoury. Moreover, the exact state of information of the other side is unknown. As we like to study real life situations by game theoretical means the above argument emphasizes the importance of a general theory about games with incomplete information.

Harsanyi (1967-1968) developed a model on which most of the study on incomplete information is based. His "Bayesian analysis for games with incomplete information" was only recently formalized by Mertens and Zamir (1985). Results not using this approach can be found e.g. in Megiddo (1980).

For studying the role of "learning" and "threats" in an incomplete information environment, repeated games (= multistage games) seem an appropriate framework. Here, the players have to make a decision several times which leaves room to statistical inferences and "threat-behaviour". Very important work was done by Aumann and Maschler (1966-1968) who laid the foundation to further research in this area.
In section 2 we try to give an impression of the results and problems in

the study of these so-called repeated games with incomplete information. Throughout this section we assume that

(1) only two players are involved in the game, player 1 and player 2.

(2) the same game is being played repeatedly.

Under these assumptions we present a general model which covers various cases of incomplete information in repeated games.

First we discuss the situation in which incomplete information is on the part of one player only (lack of information on one side) and where the players are in conflict, i.e. the gains of player 1 are equal to the losses of player 2 and vice versa (the zero-sum case). In this survey, some basic techniques and ideas in the study on incomplete information are illustrated. Subsequently it is shown that some of the results for lack of information on one side can not be generalized for a natural extension : the zero-sum case with lack of information on both sides. These results together with extensions can be found in the works of Aumann and Maschler (1966-1968) , Mertens and Zamir (1971-1972,1976,1980), Kohlberg (1975), Sorin (1980) and others. After that, the attention is directed to the non zero-sum case. Here only the case with lack of information on one side has been studied. Although many of the basic ideas used in the zero-sum case can be helpful in this case too, the results look quite different. New phenomena arise and pose new difficulties. In this part we closely follow Hart (1985).

In a somewhat different context section 3 describes a first attempt to evaluate and compare various information types. A different kind of information may lead to a different behaviour. How does a change in the state of information affect the payoffs ? The study is restricted to relatively simple one shot games, in which the players has to decide only once. This section subscribes to works of Levine and Ponssard (1977) and Borm (1987).

2. REPEATED GAMES OF INCOMPLETE INFORMATION

In this section we concentrate on an aspect of information evaluation which involves questions like

- How can I get the maximal profit out of my information ?

- What information do I reveal by taking a certain action ?
- Can my information be helpful to force a desired outcome ?

In this context terms like "learning process", "exchange of information", "threats" and "punishments" come in.

A general model for repeated games with incomplete information is presented in 2.1. It is argued that most of the games studied so far fit into this model. 2.2 discusses the zero-sum case, 2.3 the non zero-sum case.

2.1 The model

The classes of repeated games we study are given by the following

(i) Two players, player 1 and player 2.

(ii) A finite set M_1 of choices for player 1 and a finite set M_2 of choices for player 2. $M_1 := \{1,\dots,m_1\}$, $M_2 := \{1,\dots,m_2\}$.

(iii) A finite set K of stage games. To each $k \in K$ there corresponds a pair of $m_1 \times m_2$- matrices (A^k, B^k).

$$A^k = [A^k_{ij}]_{i \in M_1, j \in M_2}, \quad B^k = [B^k_{ij}]_{i \in M_1, j \in M_2}, \quad |K| = r$$

We often identify K with $\{1,\dots,r\}$}.

(iv) A probability vector $p = (p_1,\dots,p_r) \in \Delta_r := \{p \in \mathbb{R}^r \mid \Sigma_{k=1}^r p_k = 1,$ $p_k \geqq 0$ for $k \in \{1,\dots,r\}\}$.

Without loss of generality we assume $p \in \Delta_r^+ := \{p \in \Delta_r \mid p_k >$ for $k \in \{1,..,r\}\}$.

(v) Two partitions of K, K^1 for player 1 and K^2 for player 2 :

$K^1 = \{K^1(1),\dots,K^1(r_1)\}$ $\quad (r_1 \in \mathbb{N},\ r_1 \leqq r)$

$K^2 = \{K^2(1),\dots,K^2(r_2)\}$ $\quad (r_2 \in \mathbb{N},\ r_2 \leqq r)$

(vi) An element x of K is chosen according to the probability vector p.

(vii) Before stage 1 player i ($i \in \{1,2\}$) is informed about the $1_i \in \{1,\dots,r_i\}$ with $x \in K^i(1_i)$. A subjective probability vector $\rho^i(1_i)$ over K is made accordingly, i.e.

$$\rho^i(1_i) \in \Delta_r,\ \rho^i_k(1_i) = \Big(\sum_{j \in K^i(1_i)} p_j\Big)^{-1} \cdot p_k \text{ if } k \in K^i(1_i)$$
$$= 0 \qquad \text{else}$$

(viii) At each stage $m = 1,2,\dots$ player 1 chooses an element $i_m \in M_1$ and player 2 chooses an element $j_m \in M_2$. This is done simultaneously, i.e. without either player knowing what the other did. Player 1 and player 2 get the payoffs $A_{i_m j_m}$ and $B_{i_m j_m}$ respectively, but they don't observe these payoffs.

(ix) After each stage m = 1,2,... player i ($i \in \{1,2\}$) receives a signal as a function of x and the choices made in stage m. In this way we define signalfunctions λ_1 and λ_2 :

$$\lambda_i : K^i \times M_1 \times M_2 \to L_i \qquad (i \in \{1,2\})$$

where L_1 and L_2 are the signalsets of player 1 and player 2, respectively.

(x) Both players have perfect recall (i.e. they don't forget what they are told in the previous stages).

(xi) All of (i) - (x) is common knowledge to both players. Especially, player 2 knows $K^1, K^2, \rho^1(.), \lambda_1, \ldots$.

Games based on (i) - (xi) are called finite if the number of stages is finite, infinite otherwise. Finite games will be denoted by $G_n(p)$ with n being the number of stages, infinite games by $G_\infty(p)$. Note that by varying the partitions in (v) and the signalfunctions in (ix) the model can cover several types of games with incomplete information.

Following Harsanyi, these games with incomplete information can be equivalently viewed as games with complete but imperfect information. Here the uncertainty players have is not about the rules of the game (e.g. payoffs) but only about the moves previously made (by the players or by chance). This is accomplished by adding an extra stage m = 0, at which "nature" chooses $x \in K$ according to the probability vector p.

Next we describe the strategy sets of the players in $G_n(p)$, $n \in \{1,2,\ldots,\infty\}$. A pure strategy σ of player 1 in $G_n(p)$ is a collection $\{\sigma_m\}_{m \in \{1,\ldots,n\}}$, where for all $m \in \{1,\ldots,n\}$

$$\sigma_m := K^1 \times H_m \to M_1$$

with $H_m := (M_1 \times M_2)^{m-1}$

i.e. for every partition element K^1_l, $l \in \{1,\ldots,r_1\}$, and every history of actions $h_m \in H_m$, σ_m determines an action $i_m \in M_1$ for stage n. For $n = \infty$ this description is not entirely correct. Formally, we have to define a pure strategy σ in $G_\infty(p)$ as a sequence $\{\sigma_m\}_{m \in \mathbb{N}}$ such that etc. However, to shorten our descriptions we leave out this distinction from now on.

A mixed strategy is, as usual, a probability distribution over the set of pure strategies. However, since $G_n(p)$, $n \in \{1,\dots,\infty\}$, is a game with perfect recall, we can restrict ourselves to behavior strategies (for $n \in \mathbb{N}$ cf. Kuhn (1953), for $n = \infty$ cf. Aumann (1964)). If behavior strategies are used the players make independent randomizations at each move. Formally, a behavior strategy σ of player 1 in $G_n(p)$ is a collection $\{\sigma_m\}_{m \in \{1,\dots,n\}}$, where for all $m \in \{1,\dots,n\}$

$\sigma_m : K^1 \times H_m \to \Delta_{m_1}$

Analogously strategies τ for player 2 are defined by replacing σ by τ, K^1 by K^2, M_1 by M_2 and m_1 by m_2.

So far we have only defined sequences of payoffs (cf (viii)). For evaluating games with incomplete information we will concentrate on the expected average of such sequences. Therefore, we look at the expectation of $\alpha_n := \frac{1}{n} \Sigma_{m=1}^n A^x_{i_m j_m}$ and $\beta_n := \frac{1}{n} \Sigma_{m=1}^n B^x_{i_m j_m}$. These expectations depend on the probability vector p and the strategies σ and τ which are used :

$\gamma_n^2(p,\sigma,\tau) := \mathbb{E}_{p,\sigma,\tau}\, \beta_n$ and

$\gamma_n^1(p,\sigma,\tau) := \mathbb{E}_{p,\sigma,\tau}\, \alpha_n$,

It may be noted that the above description is appropriate in the version with complete but imperfect information. In the incomplete information version it would be more consistent to look at r_1- and r_2-vectors of average payoffs. Once x is chosen, the partition elements of K^1 and K^2 to which x belongs are determined. At that instant the payoffs for other "types" (this terminology is due to Harsanyi) are not important anymore. Hence we look at

$w_n^1(p,\sigma,\tau) = (w_n^1(p,\sigma,\tau;1),\dots,w_n^1(p,\sigma,\tau;r_1))$ and $w_n^2(p,\sigma,\tau) = (w_n^2(p,\sigma,\tau;1),\dots,$ $w_n^2(p,\sigma,\tau;r_2))$

with $w_n^1(p,\sigma,\tau,l) := \mathbb{E}_{\rho^1(l),\sigma,\tau}(\frac{1}{n} \sum_{m=1}^{n} A^x_{i_m j_m})$ $\quad (l \in \{1,\dots,r_1\})$

and $w_n^2(p,\sigma,\tau,l) := \mathbb{E}_{\rho^2(l),\sigma,\tau}(\frac{1}{n} \sum_{m=1}^{n} B^x_{i_m j_m})$ $\quad (l \in \{1,\dots,r_2\})$.

However, for most of our aims both descriptions turn out to be equivalent and can be used interchangeable.

2.2 The zero-sum case

This paragraph deals with games based on (i)-(xi) in which all stage games are zero-sum games, i.e. $(A^k,B^k) = (A^k,-A^k)$ for all $k \in K$. We will restrict our attention to two fundamental and relatively simple classes : standard information with lack of information on one side and on both sides, respectively. This is done because even in these cases things get complicated and were not fully understood for a long time. Besides, basic ideas and methods can be better understood here. We briefly survey the most important results and illustrate them by giving an example.

Other types of games with incomplete information can be fitted into the model of 2.1 too, like a generalisation of the standard information case by means of information matrices. The interested reader is referred to the works of Kohlberg and Zamir (1974) and Mertens and Zamir (1977). Repeated sequential games with incomplete information, in which the players do not move simultaneously but sequentially (i.e. player 1 moves first, his action is told to player 2, player 2 makes his move), don't seem to fit into the model. However, by adjusting the strategies and signalfunctions a sequential game can be viewed as a special kind of a simultaneous game. The results found for simultaneous games also hold for sequential games but can be strengthened considerably. For a more detailed description we refer to Ponssard and Zamir (1973), Ponssard (1975) and Sorin (1980).

2.2.1 Standard information : lack of information on one side

We consider games $G_n(p)$, $n \in \{1,\dots,\infty\}$ and $p \in \Delta_r$, with $K^1 = \{\{1\},\{2\},\dots,\{r\}\}$ and $K^2 = \{\{1,\dots,r\}\}$. Hence $r_1 = r, r_2 = 1$. To fit the model we assume that $K^1(l) = \{l\}$ for all $l \in \{1,\dots,r\}$. Then $\rho^1(l) = 1$ for all $l \in \{1,\dots,r\}$ and $\rho^2(l) = p$.

What this means is that player 1 exactly knows what game is being played repeatedly, whereas player 2 only knows the probability p_k of the game (A^K,B^K) being played. This is what is called lack of information on one side. Furthermore we assume the following

$L_1 = L_2 = M_1 \times M_2$ and

$\lambda_1(\{l\},i.j) = (i,j)$, $\lambda_2(\{1,\dots,r\},i,j) = (i,j)$ $(l \in \{1,\dots,r\}$, $i \in M_1$,

$j \in M_2$). This is called standard information. After each stage the actions taken in that stage are announced to both players.

Because $|K^2| = 1$ we may consider a behavior strategy τ of player 2 as a collection $\{\tau_m\}_{m \in \{1,\dots,n\}}$ where for all m

$$\tau_m : (M_1 \times M_2)^{m-1} \to \Delta_{m_2}.$$

Because $|K^1(l)| = 1$ for all $l \in \{1,\dots,r_1\}$ a behavior strategy σ of player 1 can be described by an r-vector $(\sigma^1,\dots,\sigma^r)$ with σ^k being a collection $\{\sigma^k_m\}_{m \in \{1,\dots,n\}}$, $k \in \{1,\dots,r\}$, where for all m

$$\sigma^k_m : (M_1 \times M_2)^{m-1} \to \Delta_{m_1}.$$

Further we define for each $k \in \{1,\dots,n\}$, $m \in \{1,\dots,n\}$ and $i \in M_1$

$$\sigma_m := (\sigma^1_m,\dots,\sigma^r_m) \text{ and } \sigma^k_m(i) := (\sigma^k_m)_i.$$

Since for the zero-sum case $\alpha_n = -\beta_n$ for all $n \in \mathbb{N}$ we can concentrate on

$$\gamma_n(p,\sigma,\tau) := \mathbb{E}_{p,\sigma,\tau}\left(\frac{1}{n}\sum_{m=1}^{n} A^x_{i_m j_m}\right)$$

the expected n-stage average payoff to player 1 if σ and τ are used. Note that for finite games $G_n(p)$ we have a finite number of pure strategies. For player 1 this number is equal to m_1^a, for player 2 to $m_2^{r^{-1}.a}$, with $a = r.\sum_{j=0}^{n-1}(m_1.m_2)^j$. This can be seen by counting the information sets in the version with complete but imperfect information. So, using the minimax criterium, we may consider the (minimax) value $v_n(p)$ of $G_n(p)$ defined by

$$v_n(p) = \min_\tau \max_\sigma \gamma_n(p,\sigma,\tau) = \max_\sigma \min_\tau \gamma_n(p,\sigma,\tau).$$

It appears that v_n is a concave function on Δ_r, i.e.

$$v_n(\mu p^1 + (1-\mu)p^2) \geqq \mu v_n(p^1) + (1-\mu)v_n(p^2) \quad \text{for all } \mu \in [0,1] \text{ and}$$
$$p^1 p^2 \in \Delta_r.$$

The main result for finite games is the (forward) recursive formula for $v_n(p)$ given in theorem 1. This formula was first derived by Aumann and Maschler (1966-1968) and formally proved in Mertens and Zamir (1971-1972). Recently an alternative proof was provided by Armbruster (1983, cf Sorin (1986)).

Theorem 1. For all $p \in \Delta_r$ and $n \in \{0,1,2,\dots\}$ we have

$$v_{n+1}(p) = \max_{\sigma_1}\{\min_{\tau_1} \Sigma_{k=1}^{r} p_k \cdot \sigma_1^k A^k \tau_1 + n \Sigma_{i=1}^{m_1} \bar{\sigma}_1(i) \cdot v_n(p^2(i))\} \qquad (1)$$

where

$$\begin{cases} \bar{\sigma}_1(i) := \Sigma_{k=1}^{r} p_k \sigma_1^k(i) & (i \in M_1) \\ p^2(i) \in \Delta_r \text{ such that } p_k^2(i) = \frac{1}{\bar{\sigma}(i)} \cdot p_k \cdot \sigma_1^k(i) & (i \in M, k \in K) \end{cases}$$

Note that if player 1 uses strategy σ_1 (for stage 1), $\bar{\sigma}_1(i)$ denotes the total probability of $i \in \{1,\dots,m_1\}$ being chosen in stage 1. Theorem 1 reveals some important characteristics of games with incomplete information. By the minimax criterium an optimal strategy $\hat{\sigma}$ for player 1 in $G_n(p)$ has to guarantee $v_n(p)$ even if player 2 knows this strategy. Knowing $\hat{\sigma}$ and the "real" actions of player 1 in the previous stages too, player 2 is able to deduce a sequence of posterior probability vectors $p^1, p^2, \dots$ on K by using Bayes'law. ($p^1 := p$). It is easily verified that this sequence of random variables $\{p^m\}_{m \in \{1,\dots,n\}}$ form a martingale, i.e.

$$E_{\sigma,\tau}(p^m \mid p^1,\dots,p^{m-1}) = p^{m-1} \quad \text{for all } m \in \{1,\dots,n\}.$$

Consequently $E_{\sigma,\tau}\, p^m = p$. In some sense these posterior probability vectors measure the information which is revealed to player 2. To be exact, $E_{\sigma_m}(p^{m+1} - p^m \mid p^m)$ measures the amount of information being revealed in stage m by playing σ_m in that stage. Using the sequence $\{p^m\}_{m \in \{1,\dots,n\}}$ it is also seen that

$$\gamma_n(p,\sigma,\tau) = \frac{1}{n} \Sigma_{m=1}^{n} q_m(\sigma_m,\tau_m)$$

if $q_m(\sigma_m,\tau_m)$ denotes the expected payoff at stage m according to p^m.

Now we are able to give an interpretation of theorem 1. The first term in equation (1) represents the payoff to player 1 in the first stage of $G_{n+1}(p)$, the second term his payoff in the remaining n stages. It is seen that this last payoff depends heavily on the strategy chosen in the first stage and the information revealed there. This interaction makes it impossible to analyze the situation backwards. In this way we can not use a tool which has proved to be very hulpful in the treatment of stochastic games.

A consequence of theorem 1 is that $v_n(p)$ for each $p \in \Delta_r$ is decreasing in n. Intuitively this result is clear because in a game with more stages player 2 has more opportunity to learn and to profit from this learning.

Theorem 2 and 3 deal with two approaches for repeated games with incomplete information having a large number of stages : "limit of value" and "value of limit". "Limit of value" means that we consider the value of a finite n-stage game and let n tend to infinity. "Value of limit" means that we in $G_\infty(p)$ define a value using some kind of limiting average of the payoffs. However, for games with standard information and lack of information on one side both approaches turn out to be equivalent.

Before starting theorem 2 about "limit of value" we have to define the following. Let $\Delta(p)$ be the matrix game determined by the matrix $A(p) := \Sigma_{k=1}^{r} p_k A^k$. Then $\Delta(p)$ corresponds to the game in which player 1 ignores his information (i.e. $\sigma_m^k = \sigma_m$ for all $k \in \{1,\ldots,r\}$ and $m \in \{1,\ldots,n\}$ and consequently $p^m = p$ for all m). We therefore call $\Delta(p)$ the non-revealing game corresponding to $G_n(p)$, $n \in \{1,2,\ldots,\infty\}$, and denote its value by u(p). Note that u is continuous on Δ_r.
It follows that $v_n(p) \geqq \text{Cav } u(p)$ for all $p \in \Delta_r$ and $n \in \mathbb{N}$, where Cav u denotes the least concave function that is greater or equal to u on Δ_r, and with minor abuse of notation Cav u(p) denotes its value at p. In this way we come to the conclusion that there is a function $v : \Delta_r \to \mathbb{R}$ such that

(a) $v_n \to v$ $(n \to \infty)$, uniformly

(b) v continuous and concave

(c) $v(p) \geqq \text{Cav } u(p)$ for all $p \in \Delta_r$.

Aumann and Maschler (1966-1968) proved that v = Cav u.

Theorem 2. (i) $\lim_{n\to\infty} v_n(p) = \text{Cav } u(p) \quad (p \in \Delta_r)$

(ii) There is a $N \in \mathbb{R}$, $N > 0$ such that for all $p \in \Delta_r$

$$0 \leqq v_n(p) - \text{Cav } u(p) \leqq \frac{N}{\sqrt{n}} .$$

Their proof is based on a relation between the profit that is made by playing in a revealing way and the amount of information that is thus being revealed.

Zamir (1971-1972) proved that $O(\frac{1}{\sqrt{n}})$ is the best uniform upper bound for v_n - Cav u by giving an example in which u = Cav u and $v_n(p) \geqq \frac{p(1-p)}{\sqrt{n}}$ for all $n \in \mathbb{N}$ and $p \in \Delta_r$. Here a link can be laid to the Central Limit Theorem. If n = Cav u player 1 has to ignore his information. This leads to playing the same mixed strategy at each stage. Consequently there will be n random variation in the average payoff. According to the Central Limit Theorem this variation will be of the order $\frac{1}{\sqrt{n}}$. The example of Zamir can now be thought of as an example in which player 1 can take advantage of this natural variation by slightly deviating from the "optimal" mixed strategy. However, it is interesting to note that such kind of advantageous behavior is not always possible even if u = Cav u. An interesting point was made by Mertens and Zamir (1976) who showed that in the example of Zamir the normal distribution explicitly comes in : the limit of $\sqrt{n} \cdot v_n(p)$ is the standard normal density function evaluated at its p-quantile.

Now we come to the "value of limit" approach. Aumann and Maschler (1967) pointed out that the expectation of $\lim_{n\to\infty} \frac{1}{n} \Sigma_{m=1}^{n} A^{x}_{i_m j_m}$ may fail to exist. In this way an obvious candidate for a payoff function in $G_\infty(p)$ is ruled out. To overcome this difficulty we define the value of $G_\infty(p)$ directly without first defining payoffs.

Definition. $G_\infty(p)$ is said to have a value $v_\infty(p)$ if for all $\varepsilon > 0$ there are strategies σ_ε of player 1 and τ_ε of player 2 and an integer $N_\varepsilon \in \mathbb{N}$ such that

(i) $\gamma_n(p,\sigma_\varepsilon,\tau) \geqq v_\infty(p) - \varepsilon \qquad \forall n \geqq N_\varepsilon,\ \forall\tau$

(ii) $\gamma_n(p,\sigma,\tau_\varepsilon) \leqq v_\infty(p) + \varepsilon \qquad \forall n \geqq N_\varepsilon \ \ \forall\sigma.$

This definition implies that player 1 (player 2) can get the lim inf (lim sup) of the expected n-stage average payoff to player 1 as close to $v_\infty(p)$ as he wishes. Therefore we shall use the following terminology : $\sigma_\varepsilon(\tau_\varepsilon)$ ε-guarantees $v_\infty(p)$ in $G_\infty(p)$, or, $\sigma_\varepsilon(\tau_\varepsilon)$ is ε-optimal.

The following theorem which is due to Aumann and Maschler (1966-1968) states that "value of limit" and "limit of value" coincide (cf. theorem 2).

Theorem 3. For all $p \in \Delta_r$ we have

$v_\infty(p) = \text{Cav } u(p)$.

An intuitive approach to the proof of theorem 3 is given by Aumann (1981, 23-24) and is shortly recited below because it reflects the characteristic way of reasoning in games with incomplete information.

In a zero-sum situation information can not hurt you, so v_∞ has to be concave. Next we have that $v_\infty \geqq u$ because player 1 can choose to ignore his information. So it follows that $v_\infty \geqq \text{Cav } u$ on Δ_r.
To "prove" the opposite inequality, note that player 1 (by definition) has a strategy σ_ε which ε-guarantees v_∞. Therefore he may as well announce this strategy (and use it). Given σ_ε player 2 is able to deduce a sequence of posterior probabilities $\{p^m\}_{m \in \mathbb{N}}$ which are random variables depending on the pure actions chosen by player 1 in the various stages. This sequence is a martingale. Furthermore, because this sequence of probabilities is conditioned on more and more information there is a random variable q such that $p^m \to q$ $(m \to \infty)$ with probability 1. This means that after a finite number of stages player 1 has revealed about all the information he is ever going to reveal. From that stage on he has to play (almost) nonrevealing and the posterior probabilities will be close to q. Therefore, player 1 on average can not do better than $\mathbb{E}\, u(q)$. Using Jensen's inequality and the fact that $\mathbb{E}\, q = p$ ($E\, p^m = p$ for all m and $p^m \to q$ a.s.) we have

$v_\infty(p) \leqq \mathbb{E}\, u(q) \leqq \mathbb{E}\, \text{Cav } u(q) \leqq \text{Cav } u(\mathbb{E}\, q) = \text{Cav } u(p)$.

However, all the reasoning above is just imaginary. Player 2 can not really assume that player 1 is using any particular strategy at all and therefore will not be computing posterior probabilities after all. He has to find an ε-optimal strategy τ_ε in a totally different way. Two such strategies were provided by Aumann and Maschler (1966-1968). However, the first one is of little practical use because it makes use of the optimal strategies $\hat{\tau}_n$ for all finite games $G_n(p)$, and these are not computed that easily. In this sense the second one was more applicable, being constructed with the aid of Blackwell's theorem about repeated games with vector payoffs, and therefore called a Blackwell strategy (cf. Blackwell (1956)). In proposition 5 this strategy is described. For the sake of completeness in proposition 4 we give an ε-optimal strategy for player 1. It may be noted that both strategies have stronger properties than just ε-optimality.

Proposition 4. Let $p \in \Delta_r$ and $\hat{\sigma}$ as below. Then we have

$$\gamma_n(p,\hat{\gamma},\tau) \geq \text{Cav } u(p) \qquad \forall n \in \mathbb{N} \quad \forall \tau.$$

For $\hat{\sigma}$ determine (Carathéodory) :

$p_1,\ldots,p_r \in \mathbb{R}$ and $q^1,\ldots,q^r \in \Delta_r$ such that

(a) $\text{Cav } u(p) = \Sigma_{k=1}^{r} p_k\, u(q^K)$

(b) $p = \Sigma_{k=1}^{r} p_k\, q^K$

(c) $p_k \geq 0 \quad (k \in \{1,\ldots,r\}), \quad \Sigma_{k=1}^{r} p_k = 1$

$\hat{\sigma}$: Use in every stage strategy ξ^k, which is an optimal strategy in $\Delta(q^k)$, with probability $p_k \cdot \frac{q_x^k}{p_x}$, $k \in \{1,\ldots,r\}$.

Note that in playing $\hat{\sigma}$ player 1 once performs a lottery, in the beginning, and play stationary from then on.

Proposition 5. Let $p \in \Delta_r$ and $\hat{\tau}$ as below. Then we have :

For each $\varepsilon > 0$ there is an integer $N_\varepsilon \in \mathbb{N}$ such that

$$\gamma_n(p,\sigma,\hat{\tau}) \leq \text{Cav } u(p) + \varepsilon \qquad \forall n \geq N_\varepsilon \quad \forall \sigma.$$

For $\hat{\tau}$ determine (Cav u concave and continuous) :

$\eta \in \mathbb{R}^r$ such that

(a) $\text{Cav } u(p) = \eta \cdot p$

(b) $\text{Cav } u(q) \leq \eta \cdot q \qquad$ for all $q \in \Delta_r$.

Let $S := \{z \in \mathbb{R}^r \mid z_k \leq \eta_k$ for each $k \in \{1,\ldots,r\}\}$ and define $x_n \in \mathbb{R}^r$ as the average vector payoff to player 1 after stage n-1 ($n \in \{2,3,\ldots\}$).
Note that $x_n \in V := \{z \in \mathbb{R} \mid z_k \leq \max\{\,|A_{ij}^1|\,;\ i \in M_1, j \in M_2,\ 1 \in K\}\}$.
Let y_n be such that $d(x_n,y_n) = \min_{y \in S} d(x_n,y)$ (Euclideau metric d) y_n exists because $S \cap V \neq \emptyset$).
Further, if $x_n \notin S$ we define $1^n \in \Delta_r$ by

$$1_k^n := \frac{(x_n)_k - (y_n)_k}{\Sigma_{1=1}^{r}((x_n)_1 - (y_n)_1)} \quad , \qquad k \in \{1,\ldots,r\}.$$

Let Θ^n be an optimal strategy for player 2 in $\Delta(1^n)$

$\hat{\tau}$: In stage n, play

$$\begin{cases} \text{arbitrary} & \text{if } n = 1 \text{ or } x_n \in S \\ \Theta^n & \text{else} \end{cases}$$

2.2.2 Standard information : lack of information on both sides

We consider games $G_n(p)$ of 2.1 with

$K^1 = \{K^1(1),\dots,K^1(\Gamma)\}$ and $K^2 = \{K^2(1),\dots,K^2(\nu)\}$ for Γ and ν such that

$|K^1(s)| = \nu$ for all $s \in \{1,\dots,\Gamma\}$

$|K^2(t)| = \Gamma$ for all $t \in \{1,\dots,\nu\}$ (Hence $\Gamma \cdot \nu = r$).

Consequently, K can be arranged in a $\Gamma \times \nu$-matrix of games such that the elements of K^1 form the rows and those of K^2 the columns. Accordingly, we may provide the stage games with a double labeling : $A^{s,t}$ ($s \in \{1,\dots,\Gamma\}$, $t \in \{1,\dots,\nu\}$).

Above assumptions imply that we are in the consistent case as defined by Aumann and Maschler (1967). It is therefore allowed to restrict our attention to the so-called "independent case" because any game meeting the consistency requirement, is equivalent to a game meeting the independency requirement by adjusting the probability vector p and the stage matrices A^k, $k \in \{1,\dots,r\}$.

For the independent case we assume the probability of the (s,t)-element of K, $s \in \{1,\dots,\Gamma\}$ and $t \in \{1,\dots,\nu\}$, to be $\pi_s \cdot \tilde{\pi}_t$, where $\pi = (\pi_1,\dots,\pi_\Gamma)$ and $\tilde{\pi} = (\tilde{\pi}_1,\dots,\tilde{\pi}_\nu)$ are two probability vectors : $\pi \in \Delta_\Gamma$, $\tilde{\pi} \in \Delta_\nu$.

As a motivation for the term "independent case" we can say that player 1's conditional probability on the "types" of player 2 is independent of his own "type" (similarly for player 2).

To emphasize that we assume the independent case we will use the notation $G_n(\pi,\tilde{\pi})$ instead of $G_n(p)$.

Furthermore we assume that λ_1,λ_2, L_1 and L_2 are like in 2.2.1, i.e. such that the players have standard information. It may be noted that games with standard information and lack of information on one side can be fitted into the model too : let $\Gamma = r$ and $\nu = 1$, the independency requirement is fulfilled trivially.

A behavior strategy σ of player 1 in $G_n(\pi,\tilde{\pi})$ can be described by a

Γ-vector $(\sigma^1,\ldots,\sigma^\Gamma)$ where σ^s is a collection $\{\sigma^s_m\}_{m \in \{1,\ldots,n\}}$ with

$$\sigma^s_m : (M_1 \times M_2)^{m-1} \to \Delta_{m_1} \quad , \qquad s \in \{1,\ldots,\Gamma\}.$$

For $m \in \{1,\ldots,n\}$, $s \in \{1,\ldots,\Gamma\}$ and $i \in \{1,\ldots,m_1\}$ we define

$$\sigma_m := (\sigma^1_m,\ldots,\sigma^\Gamma_m) \text{ and } \sigma^x_m(i) := (\sigma^k_m)_i .$$

Analogous definitions can be given for a behavior strategy τ for player 2. Again we concentrate on

$$\gamma_n(\pi,\tilde{\pi},\sigma,\tau) := \mathbb{E}_{\pi,\tilde{\pi},\sigma,\tau}\left(\frac{1}{n} \cdot \Sigma^n_{m=1} A^x_{i_m j_m}\right)$$

and

$$v_n(\pi,\tilde{\pi}) := \min_\tau \max_\sigma \gamma_n(\pi,\tilde{\pi},\sigma,\tau) = \max_\sigma \min_\tau \gamma_n(\pi,\tilde{\pi},\sigma,\tau).$$

For finite games $G_n(\pi,\tilde{\pi})$, the results are very similar to those found for the case with lack of information on one side. The proofs given there can be generalized almost directly. These results can be found in Mertens and Zamir (1971-1972) and are summarized in theorem 6.

Theorem 6. For all $n \in \mathbb{N}$, $\pi \in \Delta_\Gamma$ and $\tilde{\pi} \in \Delta_\nu$ we have

(i) $v_n(.,\tilde{\pi})$ is a concave function on Δ_Γ

$v_n(\pi,.)$ is a convex function on Δ_ν

(a function v is convex iff -v is concave)

(ii) the recursive formula

$$v_n(\pi,\tilde{\pi}) = \frac{1}{n} \max_{\sigma_1} \min_{\tau_1} \{\Sigma^\Gamma_{s=1} \ \Sigma^\nu_{t=1} \ \pi_s \ \tilde{\pi}_t \ \sigma^s_1 \ A^{s,t} \ \tau^t_1 + (n-1) . \ \Sigma^{m_1}_{i=1} \ \Sigma^{m_2}_{j=1} \ \bar{\sigma}_1(i) .$$
$$\bar{\tau}_1(j) \ . \ v_n(\pi^2(i),\tilde{\pi}^2(j))\}$$

where

$$\bar{\sigma}_1(i) := \Sigma^\Gamma_{s=1} \pi_s . \ \sigma^s_1(i) \ , \ \bar{\tau}_1(j) = \Sigma^\nu_{t=1} \tilde{\pi}_t . \ \tau^t_1(j) \quad (i \in M_1, \ j \in M_2)$$

and

$\pi^2(i) \in \Delta_\Gamma$, $\tilde{\pi}^2(j) \in \Delta_\nu$ such that for each $s \in \{1,\ldots,\Gamma\}$ and $t \in \{1,\ldots,\nu\}$

$$\pi^2_s(i) = \frac{1}{\bar{\sigma}_1(i)} \ . \ \pi_s . \ \sigma^s_1(i), \ \tilde{\pi}^2_t(j) = \frac{1}{\bar{\tau}_1(j)} \ . \ \tilde{\pi}_\tau \ . \ \tau^t_1(j).$$

Again we see the reliance on posterior probability vectors. For example, if we assume that player 1 knows the exact strategy of player 2 in the first stage, he can after stage 1 compute the conditional probability of player 2 being of "type" $t \in \{1,\ldots,\nu\}$, because he also gets informed about action j_1 of player 2. The two sequences $\{\pi^m\}_{m \in \{1,\ldots,n\}}$ and $\{\tilde{\pi}^m\}_{m \in \{1,\ldots,n\}}$ are martingales.

The more interesting part in games with lack of information on both sides is about infinite games. Here, in contrast to games with lack of information on one side, "value of limit" and "limit of value" are not equivalent. Before stating this result formally we introduce the following. The non-revealing game $\Delta(\pi,\tilde{\pi})$ corresponding to $G_n(\pi,\tilde{\pi})$, $n \in \{1,2,\ldots,\infty\}$ is defined to be the matrix game determined by the matrix $A(\pi,\tilde{\pi}) := \Sigma_{s=1}^{\Gamma} \Sigma_{t=1}^{\nu} \pi_s \tilde{\pi}_t \cdot A^{s,t}$. Its value $u(\pi,\tilde{\pi})$ is continuous on $\Delta_\Gamma \times \Delta_\nu$. The non-revealing game can be thought of as the game in which both players ignore their information.

The following theorem which is due to Mertens and Zamir (1971-1972) states that the "limit of value" approach is valid for the games with lack of information on both sides we defined above and that this limit is determined by two functional equations.

In this paragraph we, from this moment on, will assume that for a function $g : \Delta_\Gamma \times \Delta_\nu \to \mathbb{R}$, "Cav" is taken w.r.t. Δ_Γ, "Vex" w.r.t. Δ_ν. Formally,

$$\text{Cav } g := \min\{h : \Delta_\Gamma \times \Delta_\nu \to \mathbb{R};\ h(.,\tilde{\pi}) \text{ concave for all } \tilde{\pi} \in \Delta_\nu \text{ and } h(\pi,\tilde{\pi}) \geq g(\pi,\tilde{\pi}) \text{ for all } (\pi,\tilde{\pi}) \in \Delta_\Gamma \times \Delta_\nu\}.$$

$$\text{Vex } g := \max\{h : \Delta_\Gamma \times \Delta_\nu \to \mathbb{R};\ h(\pi,.) \text{ convex for all } \pi \in \Delta_\Gamma \text{ and } h(\pi,\tilde{\pi}) \leq g(\pi,\tilde{\pi}) \text{ for all } (\pi,\tilde{\pi}) \in \Delta_\Gamma \times \Delta_\nu\}.$$

Theorem 7. Let $\pi \in \Delta_\Gamma$, $\tilde{\pi} \in \Delta_\nu$. Then we have

(i) $\lim_{n\to\infty} v_n(\pi,\tilde{\pi})$ exists. $v(\pi,\tilde{\pi}) := \lim_{n\to\infty} v_n(\pi,\tilde{\pi})$.

(ii) $v(\pi,\tilde{\pi})$ is the unique simultaneous solution of the following two functional equations (a) and (b) :

(a) $w(\pi,\tilde{\pi}) = \text{Vex}\max\{u(\pi,\tilde{\pi}),w(\pi,\tilde{\pi})\}$

(b) $w(\pi,\tilde{\pi}) = \text{Cav}\min u(\pi,\tilde{\pi}),w(\pi,\tilde{\pi})\}$.

(iii) If $\text{Cav Vex } u(\pi,\tilde{\pi}) = \text{Vex Cav } u(\pi,\tilde{\pi})$ then $v(\pi,\tilde{\pi}) = \text{Cav Vex } u(\pi,\tilde{\pi})$.

The proof of (i) and (ii) can be found in Mertens and Zamir (1971-1972) or in Sorin (1980). To get acquainted with the notation we give the proof of (iii).

Proof of (iii). We first prove that Vex Cav $u(\pi,\tilde{\pi})$ is a solution of (a) in (ii). Observe that

$$\text{Vex Cav } u(\pi,\tilde{\pi}) = \text{Vex max } \{u(\pi,\tilde{\pi}), \text{ Cav } u(\pi,\tilde{\pi})\}$$
$$\geq \text{Vex max } \{u(\pi,\tilde{\pi}), \text{ Vex Cav } u(\pi,\tilde{\pi})\}.$$

Further,

$$\text{Vex Cav } u(\pi,\tilde{\pi}) \leq \max\{u(\pi,\tilde{\pi}), \text{ Vex Cav } u(\pi,\tilde{\pi})\}.$$

So Vex(Vex Cav $u(\pi,\tilde{\pi})$) $\leq$ Vex max$\{u(\pi,\tilde{\pi})$, Vex Cav $u(\pi,\tilde{\pi})\}$.

But Vex(Vex Cav $u(\pi,\tilde{\pi})$) = Vex Cav $u(\pi,\tilde{\pi})$.

Hence, Vex Cav $u(\pi,\tilde{\pi})$ is a solution of (a). Similarly we can also prove that Cav Vex $u(\pi,\tilde{\pi})$ is a solution of (b). Vex Cav $u(\pi,\tilde{\pi})$ (= Cav Vex $u(\pi,\tilde{\pi})$) is the unique simultaneous solution of (a) and (b) because for any solution of (a) and (b) :

$v(\pi,\tilde{\pi}) \geq$ Vex $u(\pi,\tilde{\pi})$ (cf. (a))

$v(.,\tilde{\pi})$ concave on Δ_Γ (cf. (b)).

Hence $v(\pi,\tilde{\pi}) \geq$ Cav Vex $u(\pi,\tilde{\pi})$.

Similarly it follows that $v(\pi,\tilde{\pi}) \leq$ Vex Cav $u(\pi,\tilde{\pi})$ □

Originally, Mertens and Zamir showed that the set of equations (a) and (b) had a unique solution by using the game-theoretical context of the problem. In 1977 the result was generalized by giving a proof which did not rely on any game-theoretical considerations.

In the following definition "min max" and "max min" of $G_\infty(\pi,\tilde{\pi})$ are introduced. By means of these concepts the "value of limit" approach is formalized (cf. Mertens and Zamir (1977)).

Definition. Let $\pi \in \Delta_\Gamma$, $\tilde{\pi} \in \Delta_\nu$. Let $\rho : \Delta_\Gamma \times \Delta_\nu \to \mathbb{R}$

(i) $\rho(\pi,\tilde{\pi})$ is called a min max of $G_\infty(\pi,\tilde{\pi})$ if

(a) $\forall\tau\ \forall\varepsilon > 0 : \exists\, N_{\tau,\varepsilon} \in \mathbb{N} : \exists\sigma_{\tau,\varepsilon}\ [\gamma_n(\pi,\tilde{\pi},\sigma_{\tau,\varepsilon},\tau) \geq \rho(\pi,\tilde{\pi}) - \varepsilon\ \forall n \geq N_{\tau,\varepsilon}]$

(b) $\forall\varepsilon > 0 : \exists\, N_\varepsilon \in \mathbb{N} : \exists\tau_\varepsilon\ [\gamma_n(\pi,\tilde{\pi},\sigma,\tau_\varepsilon) \leq \rho(\pi,\tilde{\pi}) + \varepsilon\ \forall\sigma\ \forall n \geq N_\varepsilon]$

(ii) $\rho(\pi,\tilde{\pi})$ is called a max min of $G_\infty(\pi,\tilde{\pi})$ if

(a) $\forall\sigma, \forall\varepsilon > 0 : \exists\, N_{\sigma,\varepsilon} \in \mathbb{N} : \exists\tau_{\sigma,\varepsilon}[\gamma_n(\pi,\tilde{\pi},\sigma,\tau_{\sigma,\varepsilon}) \leq \rho(\pi,\tilde{\pi}) + \varepsilon\ \ \forall n \geq N_{\sigma,\varepsilon}]$

(b) $\forall\varepsilon > 0 : \exists\, N_\varepsilon \in \mathbb{N} : \exists\,\sigma_\varepsilon[\gamma_n(\pi,\tilde{\pi},\sigma_\varepsilon,\tau) \geq \rho(\pi,\tilde{\pi}) - \varepsilon\ \forall\tau\ \forall n \geq N_\varepsilon$.

It is easy to verify that if a min max (or max min) exists it is unique. The min max (max min) can then be interpreted as the smallest (greatest) payoff player 2 (player 1) can guarantee.

If min max $G_\infty(\pi,\tilde{\pi})$ and max min $G_\infty(\pi,\tilde{\pi})$ both exist and are equal, we say that $G_\infty(\pi,\tilde{\pi})$ has a value which equals $v_\infty(\pi,\tilde{\pi})$ = min max $G_\infty(\pi,\tilde{\pi})$. Hence, if a value exists, it is unique.
The following theorem describes min max and max min and states that the "value of limit" approach is not valid for all games with lack of information on both sides.

Theorem 8. Let $\pi \in \Delta_\Gamma$, $\tilde{\pi} \in \Delta_\nu$. Then we have :
(i) Min max $G_\infty(\pi,\tilde{\pi})$ = Vex Cav $u(\pi,\tilde{\pi})$
Max min $G_\infty(\pi,\tilde{\pi})$ = Cav Vex $u(\pi,\tilde{\pi})$.
(ii) $v_\infty(\pi,\tilde{\pi})$ does not always exist.

Part (ii) of theorem 8 was firstly shown by Aumann and Maschler (1966-1968). This was done by giving an example in which Cav Vex u ≠ Vex Cav u. However, this example is a rather special one in which the "limit of value" equals Cav Vex u, i.e. $\lim_{n\to\infty} v_n$ = Cav Vex u. Later, a second example was provided by Mertens and Zamir (1971-1972), in which Cav Vex u ≠ Vex Cav u and $\lim_{n\to\infty} v_n$ is (almost everywhere) different from Cav Vex u and Vex Cav u.

2.2.3 Examples

To eludicate the results given in the previous paragraphs we in detail consider a game with lack of information on one side.
Example 1. (lack of informaiton on one side).
In the following we identify $(p,1-p) \in \Delta_2$ with $p \in [0,1]$.
Consider games $G_n(p)$, $n \in \{1,2,\ldots,\infty\}$ of 2.2.1 determined by

p / 1-p

$$A^1 = \begin{array}{c|ccc} & L & M & R \\ \hline L & 2 & 0 & 1 \\ R & 2 & 0 & -1 \end{array} \qquad A^2 = \begin{array}{c|ccc} & L & M & R \\ \hline L & 0 & 2 & -1 \\ R & 0 & 2 & 1 \end{array}$$

For these games the non-revealing game $\Delta(p)$ is given by $A(p)$ below

$$\begin{array}{cc} & \begin{array}{ccc} L & M & R \end{array} \\ \begin{array}{c} L \\ R \end{array} & \begin{bmatrix} 2p & 2(1-p) & 2p-1 \\ 2p & 2(1-p) & 1-2p \end{bmatrix} \end{array} = A(p)$$

Hence its value $u(p)$ is given by

$$u(p) = \begin{cases} 2p & \text{if} \quad 0 \leqq p \leqq 1/4 \\ 1-2p & \text{if} \quad 1/4 \leqq p \leqq 1/2 \\ 2p-1 & \text{if} \quad 1/2 \leqq p \leqq 3/4 \\ 2(1-p) & \text{if} \quad 3/4 \leqq p \leqq 1 \end{cases}$$

So we have

$$\mathrm{Cav}\, u(p) = v_\infty(p) = \lim_{n\to\infty} v_n(p) = \begin{cases} 2p & \text{if} \quad 0 \leqq p \leqq 1/4 \\ 1/2 & \text{if} \quad 1/4 \leqq p \leqq 3/4 \\ 2(1-p) & \text{if} \quad 3/4 \leqq p \leqq 1 \end{cases}$$

For determining $v_1(p)$, we look at the matrix $B(p)$ below, which exactly described $G_1(p)$. In the notation pure strategy (L,R) of player 1 means : choose L if A^1 is being played, choose R if A^2 is being played.

$$\begin{array}{cc} & \begin{array}{ccc} L & M & R \end{array} \\ \begin{array}{c} (L,L) \\ (L,R) \\ (R,L) \\ (R,R) \end{array} & \begin{bmatrix} 2p & 2(1-p) & 2p-1 \\ 2p & 2(1-p) & 1 \\ 2p & 2(1-p) & -1 \\ 2p & 2(1-p) & 1-2p \end{bmatrix} \end{array} = B(p)$$

Because (L,R) is a (weakly) dominating strategy we have

$$v_1(p) = \min\{2p, 2(1-p), 1\} = \min\{2p, 2(1-p)\}.$$

Figure 1 describes u, v_∞ and v_1

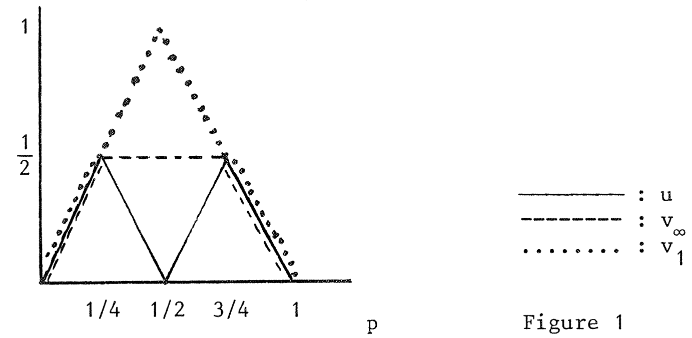

Figure 1

Optimal strategies in $G_1(p)$ are :

- for player 1 : (L,R)
- for player 2 : L if $p < 1/2$

  R if $p > 1/2$

  arbitrary if $p = 1/2$.

Using the recursion formula of theorem 1 we get

$$v_2(p) : \text{Cav } u(p) \ (= v_\infty(p)) \quad \text{for all } p \in [0,1].$$

We will only show this for $p = 1/2$. For notational convenience we introduce

$s := \sigma_1^1(L)$ (s is the probability of choosing L (according to σ) in the first stage if A' is the "real" game, $1-s = \sigma_1^1(R)$)

$q := \sigma_1^2(L)$ $(1-q = \sigma_1^2(R))$

$\tau_1 := (t_1, t_2, 1-t_1-t_2)$ $(0 \leqq t_1 \leqq 1,\ 0 \leqq t_2 \leqq 1,\ 0 \leqq t_1+t_2 \leqq 1)$.

Using this notation we get that the second stage posterior probability vector p^2 is equal to $p^2(L)$ or $p^2(R)$, with

$$p^2(L) \ (= p_1^2(L)) = \frac{s}{s+q}, \quad p^2(R) = \frac{1-s}{(1-s)+(1-q)}.$$

Substitution in the recursive formula yields (we implicitly assume $0 \leqq s \leqq 1,\ 0 \leqq q \leqq 1,\ 0 \leqq t_1 \leqq 1,\ 0 \leqq t_2 \leqq 1,\ 0 \leqq t_1+t_2 \leqq 1$)

$$v_2(\tfrac{1}{2}) = \frac{1}{2} \max_{s,q} \{\min_{t_1,t_2}\{\tfrac{1}{2}(s,1-s)\begin{pmatrix}2 & 0 & 1\\ 2 & 0 & -1\end{pmatrix}(t_1,t_2,1-t_1-t_2) + \tfrac{1}{2}(q,1-q)\begin{pmatrix}0 & 2 & -1\\ 0 & 2 & 1\end{pmatrix}.$$
$$(t_1,t_2,1-t_1-t_2)\} + (s+q)v_1(\tfrac{s}{s+q}) + ((1-s)+(1-q)).$$
$$(\tfrac{1-s}{(1-s)+(1-q)})\}$$

$$= \frac{1}{2} \max_{s,q} \{\min_{t_1,t_2} \{t_1 + (s-\tfrac{1}{2})(1-t_1-t_2) + t_2 + (\tfrac{1}{2}-q)(1-t_1-t_2)\} + \min\{s,q\} +$$
$$\min\{1-s,1-q\}\}.$$

$$= \frac{1}{2} \max_{s,q} \{\min_{t_1,t_2} \{s-q+t_1(1-s+q)+t_2(1+q-s)\} + \min\{s+1-q,q+1-s\}\}$$

$$= \frac{1}{2} \max_{s,q} \{s-q + \min\{s-q+1,q+1-s\}$$

$$= \frac{1}{2} \max_{s,q} \{\min\{2s-2q+1,1\}\} = 1/2.$$

Optimal strategies in $G_2(p)$ are :

- for player 1 : in stage 1 choose σ_1 such that $\sigma_1^1(L) \geqq \sigma_1^2(L)$ (i.e. $s \geqq q$)
  in stage 2 choose σ_2 such that $\sigma_2^1(L) = 1$, $\sigma_2^2(L) = 0$.
- for player 2 : in stage 1 choose R
  in stage 2 choose arbitrary.

Now we come to determine optimal strategies in $G_\infty(p)$.

(i) for player 1.

In case $0 \leqq p \leqq 1/4$ an optimal strategy is to choose R in every stage, because R is optimal in $\Delta(p)$ for $0 \leqq p \leqq 1/4$ and $v_\infty = \text{Cav } u = u$ on $[0,1/4]$. Similarly we see that for $3/4 \leqq p \leqq 1$ it is optimal to choose L in every stage. Thus, let $1/4 < p < 3/4$. We use proposition 4 and its notation.

So $r = 2$ and we can choose $q^1 = 1/4$, $q^2 = 3/4$, $\Gamma_1 = 3/2-2p$, $\Gamma_2 = 2p-1/2$. Then indeed

$$\rho = \Gamma_1 q^1 + \Gamma_2 q^2,\ \Gamma_1 \geqq 0,\ \Gamma_2 \geqq 0,\ \Gamma_1 + \Gamma_2 = 1.$$

$$\text{Cav } u(p) = \frac{1}{2} = \Gamma_1 \cdot \frac{1}{2} + \Gamma_2 \cdot \frac{1}{2} = \Gamma_1 \cdot u(q^1) + \Gamma_2 u(q^2).$$

An optimal strategy $\hat{\sigma}$ is therefore given by (cf. proposition 4)

With probability $\Gamma_1(\frac{1/4}{p})$, $\hat{\sigma}_m^1(R) = 1$ for all $m \in \mathbb{N}$ (R is optimal in $\Delta(p)$).

With probability $\Gamma_2(\frac{3/4}{p})$, $\hat{\sigma}_m^1(L) = 1$ for all $m \in \mathbb{N}$.

With probability $\Gamma_1(\frac{3/4}{1-p})$, $\hat{\sigma}_m^2(R) = 1$ for all $m \in \mathbb{N}$.

With probability $\Gamma_2(\frac{1/4}{1-p})$, $\hat{\sigma}_m^2(L) = 1$ for all $m \in \mathbb{N}$.

(ii) for player 2.

In case $0 \leqq p \leqq 1/4$ an optimal strategy is to choose L in every stage.
In case $3/4 \leqq p \leqq 1$ it is optimal to choose M in every stage.
So let $1/4 < p < 3/4$. We use proposition 5 and its notation.

Define $\eta = (1/2,1/2)$. Then indeed

$$\text{Cav } u(q) \leqq \eta . q = 1/2 \quad \text{for all } q \in [0,1]$$

$$\text{Cav } u(p) = \frac{1}{2} = \eta . p.$$

Now $S = \{(z_1,z_2) \in \mathbb{R}^2 \mid z_1 \leqq 1/2,\ z_2 \leqq 1/2\}$ and

$$V = \{(z_1,z_2) \in \mathbb{R}^2 \mid -2 \leqq z_1 \leqq 2,\ -2 \leqq z_2 \leqq 2\}.$$

Further we can define $x_n \in V$ as the average payoff vector after stage n-1, and $y_n \in S$ such that $d(x_n, y_n) = \min_{y \in S} d(x_n, y)$ (to get a better understanding see figure 2).
With the aid of these sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ an optimal strategy is defined exactly like is done for proposition 5.

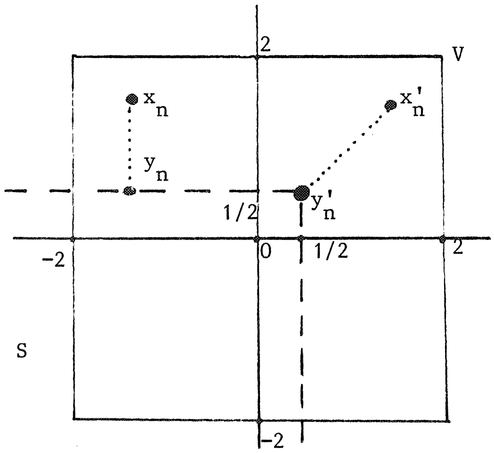

Figure 2

Example 2 is an example of Mertens and Zamir in which Cav Vex u ≠ Vex Cav u. This example is referred to at the end of 2.2.2. A detailed elaboration can be found in Mertens and Zamir (1971-1972, example 2, p. 59).

Example 2. Consider games $G_n(\pi, \tilde{\pi})$, $n \in \{1,2,...,\infty\}$, of 2.2.2 determined by the following diagram ($\Gamma = \nu = 2$, identify π with π_1) :

$$\begin{array}{cc} & \begin{array}{cc} \tilde{\pi} & 1-\tilde{\pi} \end{array} \\ \begin{array}{c} \pi \\ 1-\pi \end{array} & \begin{bmatrix} A^{1,1} & A^{1,2} \\ A^{2,1} & A^{2,2} \end{bmatrix} \end{array}$$

with

$$A^{1,1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 \end{bmatrix}, \quad A^{1,2} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A^{2,1} = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } A^{2,2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Player 1 knows whether a game $A^{1,i}$ or a game $A^{2,i}$, $i \in \{1,2\}$ is being played. Player 2 knows whether a game $A^{i,1}$ or a game $A^{i,2}$, $i \in \{1,2\}$, is being played.

2.3 The non zero-sum case

Because a thorough knowledge of the zero-sum case seems indispensable for studying the non zero-sum case, and since the zero-sum case already leads to various difficulties, there has been little research on the non zero-sum case of repeated games with incomplete information.
So far this study has been restricted to infinite games with standard information and lack of information on one side. The first approach to study this kind of games was given by Aumann and Maschler (1968). It was only recently that Sorin (1983) and Hart (1985) continued this research. Hart extended the results of Aumann and Maschler and proved some of their conjectures. In this paragraph we will briefly discuss Hart's findings.

Consider games $G_\infty(p)$ of 2.1 with lack of information on one side (i.e. $K^1 = \{\{1\},\dots,\{r\}\}$ and $K^2 = \{\{1,\dots,r\}\}$) and standard information (i.e. $\lambda_1(\{k\},i,j) = \lambda_2(\{1,2,\dots,r\},i,j) = (i,j)$ for all $k \in \{1,\dots,r\}$, $i \in M_1$ and $j \in M_2$). The stage games (A^k,B^k), $k \in \{1,\dots,r\}$, need not to be zero-sum. Let σ denote a behavior strategy of player 1, i.e. an r-vector $(\sigma^1,\dots,\sigma^r)$ with, for each $k \in \{1,\dots,r\}$, $\sigma^k = \{\sigma^k_m\}_{m \in \mathbb{N}}$ where $\sigma^k_m : (M_1 \times M_2)^{m-1} \to \Delta_{m_1}$ $(m \in \mathbb{N})$.
Similarly $\tau = \{\tau_m\}_{m \in \mathbb{N}}$, $\tau_m : (M_1 \times M_2)^{m-1} \to \Delta_{m_2}$, denotes a behavior strategy of player 2.
We recall from 2.1 that the average payoffs after stage n are given by

$$\alpha_n = \frac{1}{2}\ \Sigma_{m=1}^n A^x_{i_m j_m} \quad \text{and} \quad \beta_n = \frac{1}{2}\ \Sigma_{m=1}^n B^x_{i_m j_m} \qquad (n \in \mathbb{N})$$

and their expectations by

$$\gamma^1_n(p,\sigma,\tau) = \mathbb{E}_{p,\sigma,\tau}(\alpha_n) \text{ and } \gamma^2_n(p,\sigma,\tau) = \mathbb{E}_{p,\sigma,\tau}(\beta_n).$$

Further, since $K^2 = \{\{1,\dots,r\}\}$, we have that

$$\omega^2_n(p,\sigma,\tau) = \gamma^2_n(p,\sigma,\tau) \qquad (n \in \mathbb{N}).$$

For simplicity we therefore write $\omega_n(.)$ instead of $\omega^1_n(.)$, and because $K^1 = \{\{1\},\dots,\{r\}\}$, we can write

$$\omega_n(p,\sigma,\tau) = (\omega^1_n(\sigma,\tau),\dots,\omega^2_n(\sigma,\tau)) \qquad (n \in \mathbb{N}) \text{ where, for each}$$

$k \in \{1,\dots,r\}$,

$\omega_n^k(\sigma,\tau) := \mathbb{E}_{\sigma,\tau}(\frac{1}{n} \; \Sigma_{m=1}^{n} \; A_{i_m j_m}^{k})$

(Note that $\rho^1(k) = 1 \in \mathbb{R}$ for all $k \in \{1,\ldots,r\}$).

The non-revealing game corresponding to $G_\infty(p)$ is described by the bimatrix game $(A(p),B(p))$ where

$$A(p) := \Sigma_{k=1}^{r} \; p_k . A^k \quad \text{and} \quad B(p) := \Sigma_{k=1}^{r} \; p_k . B^k .$$

Let $u_A(p)$ ($u_B(p)$) denote the value of the zero-sum (!) game corresponding to $A(p)$ ($B(p)$).

In the following definitions (Nash)equilibria and uniform (Nash)equilibria are introduced. The definitions of a uniform equilibrium is a strengthening of the definition of a "regular" equilibrium and is suggested by the results for the zero-sum case (cf. the definitions of "minmax" and "maxmin").

Definition. $(\tilde{\sigma},\tilde{\tau})$ is an equilibrium point in $G_\infty(p)$ if

(E.1) $\liminf_{n \to \infty} \omega_n^k(\tilde{\sigma},\tilde{\tau}) \geq \limsup_{n \to \infty} \omega_n^k(\sigma,\hat{\tau})$

for all strategies σ of player 1 and $k \in \{1,\ldots,r\}$.

(E.2) $\liminf_{n \to \infty} \gamma_n^2(p,\hat{\sigma},\hat{\tau}) \geq \limsup_{n \to \infty} \gamma_n^2(p,\hat{\sigma},\tau)$

for all strategies τ of player 2.

The set of all equilibria in $G_\infty(p)$ is denoted by $E_\infty(p)$.

It is easy to verify that (E.1) is equivalent to

(E.1)* $\liminf_{n \to \infty} \gamma_n^1(p,\hat{\sigma},\hat{\tau}) \geq \limsup_{n \to \infty} \gamma_n^1(p,\sigma,\hat{\tau})$

for all strategies σ of player 1.

However, we prefer the description given in (E.1) because it is more appropriate in the incomplete information version of $G_\infty(p)$. The payoffs in an equilibrium point $(\hat{\sigma},\hat{\tau})$ are determined by taking $\sigma = \hat{\sigma}$ in (E.1) and $\tau = \hat{\tau}$ in (E.2). This leads to a vector $\bar{\omega} = (\bar{\omega}_1,\ldots,\bar{\omega}_r) \in \mathbb{R}^r$ and a number $\beta \in \mathbb{R}$ such that

$$\bar{\omega}_k = \lim_{n\to\infty} \omega_n^k(\hat{\sigma},\hat{\tau}) \qquad (k \in \{1,\ldots,r\})$$

and

$$\beta = \lim_{n\to\infty} \gamma_n^2(p,\hat{\sigma},\hat{\tau}).$$

The payoff vector $(\bar{\omega},\beta) \in \mathbb{R}^{r+1}$ corresponding to $(\hat{\sigma},\hat{\tau}) \in E_\infty(p)$ is denoted by $\eta(\hat{\sigma},\hat{\tau})$.

Definition. $(\hat{\sigma},\hat{\tau})$ is a uniform equilibrium point in $G_\infty(p)$ if

(U.1) $\liminf_{n \to \infty} \omega_n^k(\hat{\sigma},\hat{\tau}) \geqq \limsup_{n \to \infty} (\sup_\sigma \omega_n^k(\sigma,\hat{\tau}))$

for all $k \in \{1,\dots,r\}$.

(U.2) $\liminf_{n \to \infty} \gamma_n^2(p,\hat{\sigma},\hat{\tau}) \geqq \limsup_{n \to \infty} (\sup_\tau \gamma_n^2(p,\hat{\sigma},\tau))$

The set of all uniform equilibria in $G_\infty(p)$ is denoted by $U_\infty(p)$.
It is clear that every uniform equilibrium is also a "regular" equilibrium. However, the converse is not true. The difference between the two definitions becomes clear when they are translated into ε-language : for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that etc. Here, the main point is that for a uniform equilibrium this N only depends on ε but for a "regular" equilibrium it may also depend on the strategy choices.

However, the set of payoffs corresponding to equilibria and uniform equilibria coincide, as is stated in

Theorem 9. Let $G_\infty(p)$ be as above. Then we have

$$\{\eta(\sigma,\tau) \in \mathbb{R}^{r+1} \mid (\sigma,\tau) \in E_\infty(p)\} =$$
$$\{\eta(\sigma,\tau) \in \mathbb{R}^{r+1} \mid (\sigma,\tau) \in U_\infty(p)\}.$$

The proof of theorem 9 can be found in Hart (1985). It is proved together with the main result about the non zero-sum case which is given in theorem 10 below. We will restrict ourselves to its formulation in which we closely follow Hart.

Let $(A,B)_{ij} := ((A_{ij}^k)_{k \in \{1,\dots,r\}},(B_{ij}^k)_{k \in \{1,\dots,r\}}) \in \mathbb{R}^r \times \mathbb{R}^r$ $(i \in M_1, j \in M_2)$ and $F := \operatorname{conv}\{(A,B)_{ij} \mid i \in M_1, j \in M_2\} \subset \mathbb{R}^r \times \mathbb{R}^r$, where conv denotes the convex hull of a set.

F can be interpreted as the set of feasible (vector-)payoffs in the one shot game corresponding to $G_\infty(p)$. We further define

$$\Theta := \max\{\, |A_{ij}^k|, |B_{ij}^k| \mid i \in M_1, j \in M_2, k \in \{1,\dots,r\}\}$$

$$B(\Theta,m) := [-\Theta,\Theta]^m \qquad (m \in \mathbb{N})$$

Let $S \subset \mathbb{R}^r \times \mathbb{R} \times \mathbb{R}^r$ be the set consisting of all triples $(\bar{\omega},\beta,\tilde{p})$ with $\bar{\omega} \in B(\Theta,r)$, $\beta \in B(\Theta,1)$ and $\tilde{p} \in \Delta_r$ such that the following two conditions are satisfied

(S.1) $q.\bar{w} \geqq u_A(q)$ for all $q \in \Delta_r$ ($\bar{w}$ is an individually rational payoff to player 1)

$\beta \geqq \text{Vex}\,(-u_{-B}(\tilde{p}))$ (B is an individually rational payoff to player 2).

(S.2) There exist $a,b \in \mathbb{R}^r$ such that

$(a,b) \in F$, $\beta = p.b$, $\bar{w} \geqq a$ and $p\bar{w} = p.a$.

Condition (S.2) implies that $\bar{w}$ is essentially the same as a : $\bar{w}_k = a_k$ if $p_k > 0$, $k \in \{1,\ldots,r\}$. Thus, for $(\bar{w},B,\tilde{p}) \in S$ there exists a pair $(a,b) \in F$ such that $\bar{w}$ equals a in its relevant coordinates, and B is the expected payoff to player 2 corresponding to the vector-payoff b.

A justification of the term "individually rational" in (S.1) can be found in Hart (1985, p.123). Note that $-u_{-B}(p)$ is the value of the matrix game B(p), if we assume that the payoffs in the corresponding matrix are the payoffs to player 2.

The main result characterizes all equilibrium payoffs by means of the concept of an S-process, which is defined below.

Definition. Let $S = (\bar{w},B,\tilde{p}) \in B(\Theta,r) \times B(\Theta,1) \times \Delta_r$.

Let $\{S_n\}_{n \in \mathbb{N}} = \{\bar{w}_n,B_n,\tilde{p}_n\}_{n \in \mathbb{N}}$ be a sequence of random variables on an "underlying" probability space (Ω,A,P) :

$S_n : \Omega \to B(\Theta,r) \times B(\Theta,1) \times \Delta_r \quad (n \in \mathbb{N})$.

Then $\{S_n\}_{n \in \mathbb{N}}$ is called an S-process starting at s if

(P.1) $s_1 = s$ a.s.

(P.2) There exists a nondecreasing sequence $\{A_n\}_{n \in \mathbb{N}}$ of fields with finitely many elements (i.e. corresponding with a partition of Ω) to which $\{s_n\}_{n \in \mathbb{N}}$ is a martingale sequence :

$$\left.\begin{array}{l} s_n \text{ is } A_n\text{-measurable} \\ \mathbb{E}(s_{n+1} \mid A_n) = s_n \quad \text{a.s.} \end{array}\right. \qquad (n \in \mathbb{N})$$

(P.3) If $\lim_{n\to\infty} s_n = s_\infty$ a.s., then $s_\infty \in S$

(P.4) For each $n \in \mathbb{N}$: either $\bar{w}_{n+1} = \bar{w}_n$ a.s., or $\tilde{p}_{n+1} = \tilde{p}_n$ a.s.

Condition (P.1) and (P.2) imply that $\mathbb{E}s_n = s$ for each $n \in \mathbb{N}$. Furthermore, since the sequence $\{s_n\}_{n \in \mathbb{N}}$ is uniformly bounded, the Martingale Convergence Theorem, see e.g. Billingsley (1979), theorem 35.4, p.416), states that it has a.s. limit s_∞. Hence condition (P.3) is a relevant one. Condition (P.4) is such that process is a so-called bimartingale (if we "forget" the β_n-coordinate). Bimartingales are studied in Aumann and Hart (1983).

Finally, we define S^* to be the set of all points s, $s \in B(\Theta,r) \times B(\Theta,1) \times \Delta_r$, such that there exists an S-process starting at s. Having this we are able to formulate

Theorem 10. Let $\bar{w} \in \mathbb{R}^r$, $\beta \in \mathbb{R}$. Then the following two assertions are equivalent

(1) $(\bar{w},\beta) \in \{\eta(\sigma,\tau) \in \mathbb{R}^{r+1} \mid (\sigma,\tau) \in E_\infty(p)\}$

(2) $(\bar{w},\beta,p) \in S^*$.

It is easy to check that S^* is a non-empty set, but this does not imply that for every $\tilde{p} \in \Delta_r$, $G_\infty(\tilde{p})$ has at least one equilibrium point. For r = 2, however, Sorin (1983) has shown that this is the case.

The proof of theorem 10 consists of two steps :

Step 1 : $(\bar{w},\beta) \in E_\infty(p) \Rightarrow (\bar{w},\beta,p) \in S^*$

Step 2 : $(\bar{w},\beta,p) \in S^* \Rightarrow (\bar{w},\beta) \in U_\infty(p)$.

In step 2 a special class of uniform equilibria is constructed to which all "regular" equilibria are payoff equivalent. These special equilibria consist of a "master plan", which is followed by each player so long as the other does it too, and of "punishments" which are used if a deviation from the "master plan" is detected. This "master plan" is a sequence of communications between the players. Its purpose is to settle on a "desired" point in S. Without explaining it any further, we say that these communications can be of two types : "signalling" and "jointly controled lotteries". A punishment strategy is such that it keeps opponent's payoff down to an individually rationality one. This interaction between the two players is followed by means of an S-process. For a more detailed description we refer to Hart (1985).

3. ONE SHOT GAMES WITH INCOMPLETE INFORMATION

In this (short) section we pay attention to another aspect of information which is related to a question like : "Does my opponent know what information I have ?". In this context we are going to analyze and compare different types of information. This is done for the relatively simple case of two person games with only one decision-round. The style is rather informal because the purpose is just to get some understanding about different types of information. A formal approach is given in Borm (1987). The research was initiated by an article of Levine and Ponssard (1977). 3.1 describes the model of the games we will concentrate on as well as the information types we are going to analyze. The results of Borm are given in 3.2, together with an example.

3.1 Model and information types

We consider one shot games determined by the following scheme

$$\frac{1}{2} \qquad\qquad \frac{1}{2}$$

$$\begin{bmatrix} (a_{11},b_{11}) & (a_{12},b_{12}) \\ (a_{21},b_{21}) & (a_{22},b_{22}) \end{bmatrix} \quad \begin{bmatrix} (c_{11},d_{11}) & (c_{12},d_{12}) \\ (c_{21},d_{21}) & (c_{22},d_{22}) \end{bmatrix} \qquad (a_{ij},b_{ij},c_{ij},d_{ij} \in \mathbb{R})$$

$$(A^1,B^1) \qquad\qquad\qquad (A^2,B^2)$$

Interpretation : the toss of a fair coin determines whether (A^1,B^1) or (A^2,B^2) is being played. Both players have to move simultaneously and both know the description so far. If none of the players is informed about the "real" game, we will call above one shot game a game without information. We are interested in situations where one of the players (read : player 1) gets full information, i.e. he is exactly informed about which game (A^i,B^i), $i \in \{1,2\}$ is being played. We distinguish three types of such information.

(i) *Secret information* : Player 1 acquires his information secretly, which means that player 2 does not know (expect) that he is informed. Therefore, it is reasonable to assume that player 2 will have the same strategic behaviour in the game with secret information and the game without information.

(ii) *Private information* : Player 1 acquires his information in front of player 2, which means that player 2 knows that player 1 has full information. For convenience we make the following assumption : player 1 does not know (expect) that player 2 knows that he is informed. Therefore, we may assume that player 1 has the same strategic behaviour in the game with private information and the game with secret information.

Remark : Without the last assumption an infinite interaction of information could occur. Player 2 knows that player 1 has full information, but he also knows that player 1 knows this. But player 1 knows that ... etc. Such a situation could be difficult to analyze. However, this assumption is not necessary for a part of the results of Borm (1987). Here, strict dominance requirements on the bimatrix games (A^1,B^1) and (A^2,B^2) exclude an infinite interaction of information.

(iii) *Public information* : Player 1 and player 2 both acquire full information in front of each other, i.e. they know that they both have full information.

To compare these information types, we have to say something about expected payoffs in the corresponding games. In general, there could be a difficulty in determining a unique expected payoff (to each player), because even in complete information games there can be several Nash equilibria with different playoffs. However, in all the games of Levine and Ponssard (1977) and Borm (1987) the strategic behaviour of the players is (uniquely) determined. Because we will only discuss that kind of games, it is allowed to talk about "the" expected payoff to player 1 and "the" expected payoff to player 2.

Having the payoff-functions fixed (i.e. (A^1,B^1) and (A^2,B^2)) the various information types will be compared in two ways : with respect to the expected payoff of (the informed) player 1, and with respect to the expected payoff of (the uninformed) player 2. The main result is given in 3.2, but two observations can be made immediately

(a) the expected payoff to player 1 in the game with secret information is greater or equal to his expected payoff in the game without information

(b) the expected payoff to player 2 in the game with private information is greater or equal to his expected payoff in the game with secret information.

3.2 Evaluation

Having in mind observation (a) of 3.1, there can be twelve (strict) orderings of the four information types, - considering no information as an information type too -, according to the expected payoff to player 1. Similarly, because of observation (b), twelve (strict) orderings are possible according to the expected payoff to player 2.
Borm (1987) showed that all these orderings indeed do occur by introducing two special classes of games. In one of them all 12 strict orderings w.r.t. player 1 occur, in the other all 12 strict orderings w.r.t. player 2.

As an illustration of the definitions and the results mentioned above, we give an example in which for the various information types the expected payoffs to player 1 and player 2 are computed, and give an indication how the example can be extended in such a way that it is seen that all 12 strict orderings w.r.t. player 1 indeed occur.

Example 3.
Consider the following one shot game

$$\overset{\frac{1}{2}}{\begin{bmatrix} (5,4) & (3,2) \\ (-1,6) & (0,-1) \end{bmatrix}} \qquad \overset{\frac{1}{2}}{\begin{bmatrix} (1,2) & (0,3) \\ (2,1) & (\frac{1}{2},5) \end{bmatrix}}$$

$$(A^1,B^1) \qquad\qquad\qquad (A^2,B^2)$$

In the game without information the players' behaviour will be determined by the expected bimatrix game (A,B) :

$$(A,B) = \begin{bmatrix} (3,3) & (1\frac{1}{2},2\frac{1}{2}) \\ (\frac{1}{2},3\frac{1}{2}) & (\frac{1}{4},2) \end{bmatrix}$$

Hence, player 1 will choose e_1 (the first row) and player 2 will choose e_2 (the first column). Consequently the expected payoff-vector for the game without information is given by (3,3).
By assumption player 2 will also choose e_1 in the game with secret information. Player 1 is aware of this and will therefore choose e_1 if (A^1,B^1) is being played, e_2 else. Consequently, the expected payoff-vector for this case is given by $(\frac{1}{2}.5 + \frac{1}{2}.2, \frac{1}{2}.4 + \frac{1}{2}.1) = (3\frac{1}{2}, 2\frac{1}{2})$.
By assumption player 1 will have the same strategic behaviour in the game with secret information and the game with private information. Therefore player 2, in the game with private information will anticipate and will choose e_2 (instead of e_1), leading to an expected payoff-vector of $(\frac{1}{2}.3 + \frac{1}{2}.\frac{1}{2}, \frac{1}{2}.2 + \frac{1}{2}.5) = (1\frac{3}{4}, 3\frac{1}{2})$.
In case of public information both players will choose e_1 if (A^1,B^1) is the real game, and e_2 else. This leads to the expected payoff-vectors $(\frac{1}{2}.5 + \frac{1}{2}.\frac{1}{2}, \frac{1}{2}.4 + \frac{1}{2}.5) = (2\frac{3}{4}, 4\frac{1}{2})$.
In this example we see the following strict ordering of the various information types, w.r.t. player 1 : secret information is more profitable than (">") no information, no information > public information, and public information > private information.
We can get all 12 (strict) orderings by letting $x := a_{11}$, $x > 0$ and $y := c_{22}$, $0 < y < 3$, vary (cf. Borm (1987)).

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CHAPTER IV

ZERO-SUM STOCHASTIC GAMES

by Koos Vrieze

1. INTRODUCTION

In this paper we give a survey on zero-sum stochastic games. The theory of stochastic games started in 1953 with a paper of Shapley (1953). This paper was ahead of his time as will become clear below.
Zero-sum stochastic games can be regarded as extensions of two other types of decision problems, namely matrix games and Markov decision problems.

A matrix game (cf. Von Neumann, Morgenstern (1944)) is a decision problem for two players, who simultaneously and independently once have to choose an action out of a (finite) decision set. The action dependent payoffs to the players are conflicting in the sense that they add up to zero. Obviously such a decision problem can be represented by a matrix M of numbers, where the number of rows (columns) of M equals the number of actions of player 1 (player 2) and where m_{ij} equals by convention the payoff to player 1, if player 1 chooses his i-th action and player 2 his j-th action. By the rules of the game, the payoff to player 2 in this case equals $- m_{ij}$. This representation explains the term matrix game. Obviously the goal of player 1 is to maximize the outcome of the game (by choosing in an appropriate way some row of M), while player 2 tries to minimize the outcome (by choosing in an appropriate way some column of M).
A zero-sum stochastic game can be viewed as an extension of matrix games in the following sense. A stochastic game is nothing else then repeatedly playing matrix games, at well-defined discrete decision moments, according to the following rules. There are fixed a finite number, say z, of matrix games. At each decision moment the players are informed about the matrix game at hand at that moment. Next both players make a choice (simultaneously and independently) out of the action sets available in this matrix game. These choices result not only in a payoff (like in a matrix game), but also in an action dependent probability measure on the set of matrix games.

Next, according to this probability measure, a chance experiment is carried out to determine the matrix game to be played at the following decision moment. So, in stochastic games, at each decision moment, the players have as well short term as long term interests. Short term in the sense that the chosen actions determine some payoff at that moment. Long term in the sense that the actions determine the dynamic behaviour of the system at that moment, giving rise to intentions of the players of steering the system to more favourable matrix games.

As already stated, in the second place, stochastic games can be regarded as extensions of Markov decision problems (cf Denardo (1982)). A, say maximizing, Markov decision problem is defined analogously to stochastic games, with the restriction that each of the z matrix games consists of a single column. This reflects the fact that in Markov decision problems, we have to do with only one decision maker who has to choose a row out of that single column. The immediate rewards and the dynamics of the system are defined completely similar to stochastic games. Obviously, a minimizing Markov decision problem can be exposed as a stochastic game, where each of the z matrix games consists of a single row. Thus a stochastic game can be viewed as the extension of the multi-stage decision problem with one decision maker to the case of the multi-stage decision problem with two decision makers with strictly oppose interests. The theory of Markov decision problems evolved in the late fifties and the early sixties (Bellman (1957), Blackwell (1962, 1965)). At that time, people working in the field of Markov decision problems seem to be unaware of the pioneering work of Shapley. Some of the essential theorems were derived for the (simpler) case of Markov decision problems a small decade later then Shapley did for stochastic games. In this sense, Shapley was ahead of his time.

This paper is organised as follows. In section 2 the notion of stochastic game is formally introduced. Strategies are defined and three possible evaluation criteria are given.
In section 3 discounted reward stochastic games are treated. It is shown how the structural properties of these game can be discovered by combining the Shapley equations with the structural properties of matrix games.

In section 4 we look at average reward stochastic games. Puiseux series turn out to be extremely useful here. The main results are exposed, with special emphasis on optimal stationary strategies.
In section 5 total reward stochastic games are considered. In a certain sense, this criterion appears to have similar properties as the average criterion. However several problems remain open for this criterion.
In section 6 we handle a number of structured stochastic games, i.e. subclasses of stochastic games, determined by properties on the reward and transition data.
Finally in section 7 we give some approximation algorithms for computing the value and ε-optimal stationary strategies for both the discounted reward and the average reward stochastic game.

2. THE STOCHASTIC GAME MODEL

In this section we state the formal definition of a stochastic game and the way it is played.
A zero-sum stochastic game can be described by a six-tuple :
$\Gamma = \langle S,\{A_s,s \in S\},\{B_s,s \in S\},\{r_s,s \in S\},\{p_s,s \in S\},D \rangle$.
The meanings of the components are as follows.
$S := \{1,2,\ldots,z\}$, a finite set with $|S| = z$, denoting the states of the system; a state corresponds with a matrix game.
$A_s := \{1,2,\ldots,m_s\}$, for each $s \in S$, a finite set with $|A_s| = m_s$, denoting the set of available actions of player 1 in state s.
$B_s := \{1,2,\ldots,n_s\}$, for each $s \in S$, a finite set with $|B_s| = n_s$, denoting the set of available actions of player 2 in state s.
Hence the matrix game corresponding to state $s \in S$, notation M_s, has m_s rows and n_s columns.
$r_s : A_s \times B_s \to \mathbb{R}$, for each $s \in S$, a real-valued mapping on the cartesian product $A_s \times B_s$. Here $r_s(i,j)$ equals the (i,j)-the entry of M_s; r_s is called the payoff function.
$p_s : A_s \times B_s \to P(S)$, for each $s \in S$, a mapping from the cartesian product $A_s \times B_s$ on the set of probability measures on the set of states, i.e. on $P(S) := \{x = (x_1,x_2,\ldots,x_z); x_t \geq 0, \Sigma_{t=1}^{z} x_t = 1\}$. Here $(p_s(i,j))_t$ denotes the probability that the system will move to state $t \in S$, if

in state $s \in S$ player 1 chooses action $i \in A_s$ and player 2 chooses action $j \in B_s$. In the following $(p_s(i,j))_t$ will be written as $p(t \mid s,i,j)$. p_s is called the transition map.

$D := \{1,2,\ldots,N\}$ with $N \in \mathbb{N}$ or $D := \mathbb{N}$, the set of decision moments. If $D := \mathbb{N}$ then we speak of a game with infinite horizon, else of finite horizon.

In this paper we only look at infinite horizon games. At the end of this section it will be made clear that stochastic games with a finite number of decision moments can be identified with matrix games.

A stochastic game is played as follows. A starting state $s_1 \in S$ is given to both players at decision moment 1. Both players choose simultaneously and independently an action out of their respective available action sets. Say, that this results in action $i_1 \in A_{s_1}$ for player 1 and action $j_1 \in B_{s_1}$ for player 2. Then two things happen. First there is an immediate payoff $r_{s_1}(i_1,j_1)$ to player 1 from player 2 and second, the system moves to a next state according to the probability measure $p_{s_1}(i_1,j_1)$, where $p(t \mid s_1,i_1,j_1)$, for each $t \in S$, equals the probability that this next state will be state t. Then at decision moment 2 both players are informed about the new current state. Here the game proceeds as if it starts again, etc.

We assume perfect recall and complete information, i.e. at each decision moment both players perfectly remember all past states and actions that have actually occurred and both players know each function r_s and all mappings p_s completely.

As usually in non-cooperative game theory we allow the players to select at each decision moment a (pure) action according to the specification of a mixed action. Since, at each decision moment, the players have full knowledge of the history of the game up to that moment, they may use this knowledge in specifying their mixed action. Furthermore, this mixed action may depend on the stage number. Formally, let, at decision moment n, h_n be the history of the game, i.e. $h_n := (s_1,i_1,j_1,s_2,i_2,j_2,\ldots,s_{n-1},i_{n-1},j_{n-1})$, where at decision moment k, $s_k \in S$, $i_k \in A_{s_k}$ and $j_k \in B_{s_k}$ have occurred for $k = 1,2,\ldots,n-1$. Let $P(A_s)$ ($P(B_s)$) denote the set of mixed actions for player 1 (player 2) in state $s \in S$, i.e.

$P(A_s) := \{(x_1,x_2,\ldots,x_{m_s});\ x_i \geq 0 \text{ and } \Sigma_{i=1}^{m_s} x_i = 1\}$ and

$P(B_s) := \{(y_1,y_2,\ldots,y_{n_s});\ y_j \geq 0 \text{ and } \Sigma_{j=1}^{n_s} y_j = 1\}$.

Then a behaviour strategy for player 1, notation π_1, can be associated with a function π_1 on the set of triples (s,n,h_n), where $\pi_1(s,n,h_n) \in P(A_s)$ for each s,n,h_n.

Such a strategy is used as follows : if at decision moment n the state equals s and if history h_n has occurred, then player 1 chooses his pure action according to the mixed action $\pi_1(s,n,h_n)$.

A behaviour strategy for player 2, π_2, is defined analogously.

Three special types of strategies are discerned. First, a pure strategy is a strategy where $\pi_k(s,n,h_n)$ specifies with probability one some pure action for each $s \in S$, history h_n and decision moment n.

Second, a Markov strategy is a strategy where at each decision moment the mixed action only depends on the stage number and on the current state and not on the history of the game. Hence a Markov strategy for player k is a function π_k on the set of pairs (s,n), with the same interpretation as above for behaviour strategy.

Third, a stationary strategy is a strategy where at each decision moment the mixed action only depends on the current state and not on the stage number or the history of the game. For stationary strategies we introduce an apart notation, ρ for one of player 1 and σ for one of player 2. Then $\rho = \{\rho(s); s \in S\}$ with $\rho(s) \in P(A_s)$ and $\sigma = \{\sigma(s); s \in S\}$ with $\sigma(s) \in P(B_s)$ and when player 1 (player 2) decides to play a stationary strategy $\rho(\sigma)$, then each time the system is in state s he will choose his pure action according to $\rho(s)$ $(\sigma(s))$.

If both players specify a strategy, say π_1 and π_2, then for a fixed starting state $s \in S$, for each $n \in D$, this will determine a probability measure on the set of histories h_n up to decision moment n. Denote these probabilities by $P_n(s,\pi_1,\pi_2,h_n)$. From these probabilities we can derive two things.

First, since $P_{n-1}(s,\pi_1,\pi_2,.)$ can be interpreted as some marginal distribution of $P_n(s,\pi_1,\pi_2,.)$ it follows by the Kolmogorov extension theorem, that the sequence $(P_n(s,\pi_1,\pi_2,.),\ n = 1,2,...)$ can be extended to a unique probability measure on the set of infinite sequences $(s_1,i_1,j_1,s_2,i_2,j_2,...)$. Second, for each decision moment $n \in D$, the marginal distribution of the triples (s_n,i_n,j_n) occurring at decision moment n can be computed. Let $p_n(s,\pi_1,\pi_2,s_n,i_n,j_n)$ denote the probability that the triple (s_n,i_n,j_n) occurs at decision moment n if player 1 plays π_1, player 2 plays π_2 and the starting state is s. Then we can compute the expected payoffs at the decision moments. Let R(n) be the stochastic variable denoting the payoff at decision moment n, then

$$E_{s\pi_1\pi_2}[R(n)] := \sum_{s_n,i_n,j_n} p_n(s,\pi_1,\pi_2,s_n,i_n,j_n)\, r_{s_n}(i_n,j_n) \qquad (2.1)$$

Already Shapley (1953) showed that for stationary strategies this expression can be simplified considerably.
Let $E_{\pi_1\pi_2}[R(n)] := (E_{1\pi_1\pi_2}[R(n)],...,E_{z\pi_1\pi_2}[R(n)])$.
Then for a pair of stationary strategies ρ and σ it holds that

$$E_{\rho\sigma}[R(n)] = P^{n-1}(\rho,\sigma)\ r(\rho,\sigma), \qquad (2.2)$$

where $P^k(\rho,\sigma)$ equals the k-fold product of the $z \times z$-matrix $P(\rho,\sigma)$ and where the (s,t)-the element of $P(\rho,\sigma)$, notation $p(t \mid s,\rho,\sigma)$, equals :

$$p(t \mid s,\rho,\sigma) := \Sigma_{i=1}^{m_s}\ \Sigma_{j=1}^{n_s}\ p(t \mid s,i,j)\ \rho_i(s)\ \sigma_j(s) \qquad (2.3)$$

Further $r(\rho,\sigma) = (r_1(\rho,\sigma),\ r_2(\rho,\sigma),...,\ r_z(\rho,\sigma))$ with

$$r_s(\rho,\sigma) := \Sigma_{i=1}^{m_s}\ \Sigma_{j=1}^{n_s}\ r_s(i,j)\ \rho_i(s)\ \sigma_j(s) \qquad (2.4)$$

The interpretation is as follows. $p(t \mid s,\rho,\sigma)$ being the (s,t)-th element of $P(\rho,\sigma)$, equals the probability that the system moves in one step to state $t \in S$ if in state $s \in S$ player 1 plays $\rho(s)$ and player 2 plays $\sigma(s)$. It can easily be shown by induction, that the (s,t)-th element of $P^{n-1}(\rho,\sigma)$ equals the probability that at decision moment n the system is in state t if it starts at decision moment 1 in state s and if the players play the stationary strategies ρ and σ. Obviously $r_s(\rho,\sigma)$ is the expected immedate reward in state s when player 1 plays $\rho(s)$ and player 2 plays $\sigma(s)$.

Now expression (2.2) is immediate. Observe that expression (2.2) specifies simultaneously the expected payoffs at stage n for all z specific plays with starting state respectively 1,2,...,z.
Summarizing the above, we see that, associated with a pair of strategies (π_1,π_2) and a specific starting state s, there is a sequence of expected payoffs $(E_{s\pi_1\pi_2}[R(n)],\ n = 1,2,\dots)$.

In order to compare the worth of strategies, an evaluation criterion is needed, i.e. a rule which uniquely associates a real number to such a sequence. In this paper we consider three evaluation rules.

First, the discounted reward criterion, defined as

$$v^{\beta}(s,\pi_1,\pi_2) := \Sigma_{n=1}^{\infty} \beta^{n-1} E_{s\pi_1\pi_2}[R(n)] \tag{2.5}$$

Here $\beta \in (0,1)$ is the discount factor, reflecting the interest rate.

Second, the average reward criterion, defined as

$$g(s,\pi_1,\pi_2) := \liminf_{N\to\infty} \frac{1}{N} \Sigma_{n=1}^{N} E_{s\pi_1\pi_2}[R(n)] \tag{2.6}$$

Since the limit of the right-handside expression of (2.6) does not need to exist, a further specification is necessary. The choice of lim inf ("the worst case") is more or less arbitrary. However the results for average reward stochastic games do not change, when lim inf is replaced by lim sup or any convex combination of them.

Third, the total reward criterion, defined as

$$v^{T}(s,\pi_1,\pi_2) = \liminf_{N\to\infty} \frac{1}{N} \Sigma_{k=1}^{N} \Sigma_{n=1}^{k} E_{s\pi_1\pi_2}[R(n)] \tag{2.7}$$

In general this expression may be $+\infty$ or $-\infty$. We will apply this criterion to a class of stochastic games, where this expression makes sense. Observe that in case where $\Sigma_{n=1}^{\infty} E_{s\pi_1\pi_2}[R(n)]$ exists, we have $v^T(s,\pi_1,\pi_2) = \Sigma_{n=1}^{\infty} E_{s\pi_1\pi_2}[R(n)]$.

The expressions (2.5), (2.6) and (2.7) specify the expected payoffs to player 1. By definition the payoffs to player 2 are the negatives of these expressions.

As solution concept for zero-sum stochastic games the usual concept for zero-sum games in normal form is adopted. The players seek strategies which guarantee them a payoff as high as possible. Since the payoffs, for any criterion, are defined as the payoffs to player 1 (and as minus the payoffs to player 2), this concept leads to the following :
Let c(s,.,.) stand for any criterion. Then player 1 tries to find a strategy which guarantees him a payoff as close as possible to $\sup_{\pi_1} \inf_{\pi_2} c(s,\pi_1,\pi_2)$. Analogously player 2 tries to find a strategy which guarantees him an expected payoff as close as possible to $\inf_{\pi_2} \sup_{\pi_1} c(s,\pi_1,\pi_2)$. Whenever

$$c(s) := \sup_{\pi_1} \inf_{\pi_2} c(s,\pi_1,\pi_2) = \inf_{\pi_2} \sup_{\pi_1} c(s,\pi_1,\pi_2) \tag{2.8}$$

we say that the game is strictly determined for starting state s. In that case, the number c(s) is called the value of the game for starting state s. A stochastic game is said to have a value if for each starting state the game is strictly determined.

A strategy which guarantees a player the value of the game up to ε, $\varepsilon \geqq 0$, is called ε-optimal, so for player 1, π_1^* is ε-optimal if

$$\inf_{\pi_2} c(s,\pi_1^*,\pi_2) \geqq c(s)-\varepsilon \tag{2.9}$$

and π_2^* is ε-optimal for player 2 if

$$\sup_{\pi_1} c(s,\pi_1,\pi_2^*) \leqq c(s)+\varepsilon \tag{2.10}$$

A 0-optimal strategy is called optimal.

We conclude this section by a remark on stochastic games with a finite number of decision moments. For all three evaluation criteria these games can be formulated(and therefore handled)as a matrix game. Observe that for a game with finite horizon both players have a finite number of pure strategies. When we display the stochastic game for a fixed starting state as a game in extensive form, then these sets of pure strategies for the players coincide with the sets of pure strategies for the players in the extensive form game. By the results of Kuhn (1953) (cf. also the chapter by Van Damme in this book), it follows that these games can be regarded as matrix games, merely by numbering the pure strategies (being the sets of pure actions in the corresponding matrix game) and by relating to each pair of pure strategies the expected payoff as assigned by the evaluation rule to the finite stream of expected payoffs of the corresponding strategies. Hence (Von Neumann (1928)) the value of these finite horizon games exists for any criterion and both players possess optimal strategies, consisting of mixtures of pure strategies.

3. THE DISCOUNTED CRITERION

Though, in fact, Shapley considered stopping stochastic games with the total reward criterion, his 1953 paper can be signed as the start of the theory on stochastic games in general and of the discounted reward criterion especially. The mathematical techniques used in discounted reward games are similar to those used in stopping total reward games.

A stopping stochastic game is a game with the property $q(s,i,j) := 1 - \Sigma_{t=1}^{z} p(t \mid s,i,j) > 0$ for each s,i and j. $q(s,i,j)$ equals the stopping probability in state s when the players select action i and j respectively. A discounted reward stochastic game can be formulated as a total reward stopping stochastic game by adapting the transition probabilities. Take for the stopping game $1-\beta$ as the stopping probability and take $\beta p(t \mid s,i,j)$ as the probability of moving from s to t by actions i and j. It can be verified that for each pair of strategies the total reward in this stopping game equals the discounted reward in the original game.

We now state the main theorem, due to Shapley (1953), of discounted reward stochastic games. For that purpose, define for $v = (v_1,v_2,\ldots,v_z) \in \mathbb{R}^z$, for each $s \in S$, the matrix game

$$M_s^{\beta}(v) := [r_s(i,j) + \beta\, \Sigma_{t=1}^{z}\, p(t \mid s,i,j)v_t]_{i=1}^{m_s}{}_{j=1}^{n_s} \tag{3.1}$$

Further $\mathrm{Val}(M_s^{\beta}(v))$ will denote the minmax value of this game.

Theorem 3.1

(a) Discounted reward stochastic games are strictly determined.

(b) The value, say $v^\beta := (v_1^\beta, v_2^\beta, \ldots, v_z^\beta)$ equals the unique solution to the following set of functional equations :

$$v_s = \mathrm{Val}(M_s^\beta(v)), \text{ for each } s \in S \tag{3.2}$$

(c) A stationary strategy ρ for player 1 is optimal if and only if, for each $s \in S$, $\rho(s)$ is an optimal action for player 1 in $M_s^\beta(v^\beta)$. A similar result holds for player 2.

The proof of this theorem is based on the fact that the right-hand side of (3.2) represents a contraction mapping with contraction factor β on the $\mathbb{R}^z$. Hence Banach's contraction mapping theorem yields a unique solution (fixed point) to the set of equations (3.2), which turns out to be the value of the game.

An alternative proof of theorem 3.1 is given in Vrieze (1987). There the set of equations (3.2) is formulated as a non-linear programming problem (linear object function subject to quadratic constraints). Application of the Kuhn-Tucker conditions to this NLPP gives a constructive proof of all parts of theorem 3.1.

Theorem 3.1 makes in an essential way use of matrix game theory. Indeed several structural properties of discounted reward stochastic games can be shown by suitable injection of matrix game properties. For instance, by (c) of theorem 3.1 and by the Bohnenblust, Karlin and Shapley (1950) characterization of solution sets of matrix games, we derive (Vrieze and Tijs (1980)) :

Theorem 3.2.

The set of optimal stationary strategies, O_k^β, for player k, k = 1,2, in the discounted stochastic game is equal to the Cartesian product $\times_{s=1}^{z} O_k^\beta(s)$, where $O_k^\beta(s)$ is the convex polyhedron of optimal mixed actions of player k in the matrix game $M_s^\beta(v^\beta)$.

Also the Shapley-Snow (1950) results concerning the extreme optimal actions for matrix games can be extended to stochastic games (Vrieze and Tijs (1980)).

Theorem 3.3.
Let ρ be an extreme point of O_1^β and σ be an extreme point of O_2^β. Then there exists a stochastic subgame from which ρ and σ can be computed in the Shapley-Snow manner. (Here a subgame arises when pure actions are deleted from one or several states for one or both players).

Notice that this theorem gives a method, though not an efficient one, of computing the extreme optimal stationary strategies of $O_1^\beta \times O_2^\beta$ by looking at the finite number of stochastic subgames in which at each state both players have the same number of pure actions.

A next theme that lends itself to conveying matrix game properties to stochastic games, is perturbation theory. In the first place, from theorem 3.1, part (b) and the fact that the value of a matrix game is a continuous function of the entries of a matrix game, it follows that (Tijs and Vrieze (1980))

Theorem 3.4
The value of discounted reward stochastic games, considered as a function on the parameters (rewards, transitions, discount factor) is a continuous one.

Also with respect to the sets of ε-optimal stationary strategies ($\varepsilon \geqq 0$) a continuity statement can be made (Tijs and Vrieze (1980)).

Theorem 3.5
Let $O_k^\beta(\varepsilon)$ be the set of ε-optimal stationary strategies to player k, $k \in \{1,2\}$. Then $O_k^\beta(\varepsilon)$ is an upper semi-continuous multimap on the parameters of the stochastic game.

A useful implication of theorem 3.5 is the following observation. Take

$\varepsilon > 0$. Then for any two games Γ and $\tilde{\Gamma}$ which are "close enough" to each other it holds that $O_k^\beta(\varepsilon) \cap \tilde{O}_k^\beta(\varepsilon) \neq \emptyset$. Moreover it can be shown that $O_k^\beta(\varepsilon) \subset \tilde{O}_k^\beta(\varepsilon + c\,\delta)$, where c is some number determined by the parameters of Γ and δ is the distance between Γ and $\tilde{\Gamma}$.

In practical situations small deviations in the exact-values of the game parameters are inevitable. The theorems 3.4 and 3.5 show that then small changes in the game parameters induce only small changes in the solution of the game. This property increases the reliability and practicability of discounted reward stochastic games.

A last property for discounted reward stochastic games that we will mention is a topological one. Fix action spaces A_s, B_s, $s \in S$. Let SG be the class of stochastic games with these action spaces. Let USG be the subclass of SG for which both players have a unique optimal stationary strategy. Notice from theorem 3.5 that this unique optimal stationary strategy varies continuously over USG. (If a player possesses in a discounted reward stochastic game a unique optimal stationary strategy, then this strategy is his only optimal strategy, stationary or not). For a class of matrix games of fixed size, the subclass of matrix games with unique optimal actions for the both players is a dense and open subset with respect to the whole class (Bohnenblust, Karlin and Shapley (1950)). For stochastic games and analogous result, proved in an analogous way, holds (Tijs and Vrieze (1980))

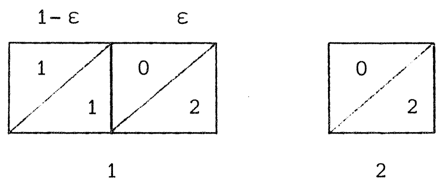
Theorem 3.6
The set USG is a dense and open subset with respect to SG.

In fact theorem 3.6 states that each open neighborhood in SG of a fixed stochastic game has a non-void intersection with the set USG. Even more it can be shown that each such neighborhood contains elements of USG that have the same value as the fixed stochastic game. Furthermore, since USG is open, it follows that a "generic" stochastic game is one belonging to USG.

4. THE AVERAGE REWARD CRITERION

Average reward stochastic games are considerably more difficult to analyse then discounted reward stochastic games. The reason can be found in the fact that the expected average reward is not continuous over the strategy space (in contrast to the discounted case).

For example :



Average reward equals 0 as long as $\varepsilon > 0$, while for $\varepsilon = 0$ the average reward equals 1 for starting state 1.

Average reward stochastic games were introduced by Gillette (1957). He considered games with perfect information (in each state one of the players has only one action available) and irreducible stochastic games (games where for each pair of stationary pure strategies (ρ,σ) the associated stochastic matrix $P(\rho,\sigma)$ (cf (2.3)) has a single ergodic class and no transient states). Blackwell and Ferguson (1968) used a slightly modified version of an example of Gillette to show for average reward stochastic games that, in general, the players need not possess optimal strategies. Even more for this example, called the big match, one of the players has no ε-optimal strategy within the class of (semi-)Markov strategies for $\varepsilon > 0$ small enough.

For a long time it was an open question whether average reward stochastic games always have a value. Only around 1980 this question was answered in the affirmative by Mertens and Neyman (1981). Before that time, results for special cases of average reward stochastic games were obtained by several authors. The emphasis was laid mainly on the existence of optimal stationary strategies for the players. Hoffman and Karp (1986) treated irreducible stochastic games. Their approach is based on results of Markov decision theory. Kohlberg (1974) analyzed so-called "repeated

games with absorbing states". These are games where all but one of the states are absorbing and where the remaining state is transient or recurrent, depending on the strategies played. The big match belongs to this class of games. Kohlberg showed that these games have a value which can be found by considering the τ-step game and letting τ tend to infinity. Later it appeared that Kohlberg's approach indicated the way in which in the general case the existence of the value can be shown.
Bewley and Kohlberg (1976, 1978) exposed in an elegant way some of the relationships between the discounted game, the τ-step game and the average reward game. Below we shall explain their use of the field of real Puiseux series.
A characterization of stochastic games with optimal stationary strategies for the both players was given in Vrieze (1987). Further, in Tijs and Vrieze (1986) it was shown that for both players there are always states which are easy to them i.e. when the game starts in such a state then the respective player can guarantee the value of the game by playing an appropriate stationary strategy.
Some of the above mentioned results will be worked out now.

We start with the introduction of Puiseux series.
Let for a positive integer M :

$$F_M := \{ \Sigma_{k=-\infty}^{K} c(k)\Theta^{k/M} \text{ ; K is an integer, } c(k) \in \mathbb{R} \text{ and such that the series } \Sigma_{k=-\infty}^{K} c(k)\tau^{k/M} \text{ converges for all sufficiently large real numbers } \tau\}.$$

Here Θ represents an arbitrarily large real number; thus the members of F_M are power series in $\Theta^{1/M}$.

Let $F := \bigcup_{M=1}^{\infty} F_M$, then it can be checked that F is an ordered field and F is called the field of real Puiseux series.
Bewley and Kohlberg (1976) extended Shapley's equations (cf (3.2)) for discounted reward stochastic games in the following way :
The set of equations (cf (3.1) and (3.2)) :

$$x_s = \text{Val}(M_s^{1/1+\Theta^{-1}}(x)) \quad \text{for each } s \in S, \tag{4.1}$$

where $x_s \in F$ and $x = (x_1, x_2, \ldots, x_z) \in F^z$, is called the limit discount equation.

That for $x \in F^2$ the right-handside of (4.1) belongs to F is a consequence of the ordered field preserving property of the value function (Weyl (1950)). Notice that (4.1) is a set of equations in the function space F^z and that for $\Theta = \tau \in \mathbb{R}$, (4.1) is equivalent to (3.1) with $\beta = \frac{1}{1+\tau^{-1}}$.

Bewley and Kohlberg (1976, 1978) proved the following:

Theorem 4.1

(a) The set of equations (4.1) has a unique solution in F^z, say $x_s^* = \Sigma_{k=-\infty}^{K} c_s(k)\Theta^{k/M}$, $s = 1,2,\ldots,z$, for which it holds that $K \leqq M$.

(b) $\Sigma_{k=-\infty}^{M} c_s(k)\tau^{k/M}$, $s = 1,2,\ldots,z$, is the value of the $\frac{1}{1+\tau^{-1}}$ discounted reward stochastic game for all τ sufficiently large.

(c) $c_s(M) = \lim_{\tau\to\infty} \tau^{-1}(\Sigma_{k=-\infty}^{M} c_s(k)\tau^{k/M}) = \lim_{\tau\to\infty} \frac{1}{\tau} FV_s(\tau)$.
Here $FV_s(\tau)$ is the total reward value of the finite horizon stochastic game with τ stages and starting state s.

(d) If player 1 has, for each $s \in S$, a real action $\rho(s)$ such that $\rho(s)$ guarantees player 1 $\text{Val}(M_s^{1/1+\Theta^{-1}}(x^*)) + O(\Theta^0)$ in the matrix game $M_s^{1/1+\Theta^{-1}}(x^*)$, then $\rho = (\rho(1), \rho(2), \ldots, \rho(z))$ is an optimal stationary strategy in the average reward game. An analogous statement holds for player 2.

Translated in terms of the discount factor β, part (c) of theorem 4.1 states that $\lim_{\beta\uparrow 1} (1-\beta) v_s^\beta$ exists for each $s \in S$ and that this limit equals the limit of the average reward values for finite horizon games with the same starting state. Later on, Neyman and Mertens (1981) showed that these limits also equal the average reward value for the infinite horizon game with the same starting state s. From part (d) one deduces immediately that stationary strategies which are uniform discount optimal are also average reward optimal. (A strategy is uniform discount optimal if it is optimal for each discount factor β close enough to 1).

The following theorem is due to Mertens and Neyman (1981) :

Theorem 4.2

(a) Average reward stochastic games hava a value.

(b) ε-optimal strategies can be constructed from the solution to the limit discount equation by computing for each stage a discount factor with the aid of rules depending on the history of the game up to that stage. Next the action at that stage with that history can be chosen as an optimal action in Shapley's equation for the computed discount factor.

The proof of this theorem is based on the results of Bewley and Kohlberg, using a martingale property.

Surely, history dependent strategies are terrible to handle. Application of stationary strategies is more favourable, since players only have to look at the current state. Below we give a characterization of stochastic games for which both players have optimal stationary strategies (Vrieze (1987)).

Theorem 4.3

Both players possess optimal stationary strategies if and only if the following set of equations has a solution $g, v(1), v(2) \in \mathbb{R}^z$:

(a) $$g_s = \operatorname{Val}_{A_s \times B_s} \left(\Sigma_{t=1}^{z} p(t \mid s,.,.) g_t\right), \quad \text{each } s \in S, \tag{4.2}$$

(b) $$v_s(1) = \operatorname{Val}_{E_s(1) \times B_s} \left(r_s(.,.) + \Sigma_{t=1}^{z} p(t \mid s,.,.) v_t(1)\right), \text{each } s \in S, \tag{4.3}$$

(c) $$v_s(2) = \operatorname{Val}_{A_s \times E_s(2)} \left(r_s(.,.) + \Sigma_{t=1}^{z} p(t \mid s,.,.) v_t(2)\right), \text{each } s \in S. \tag{4.4}$$

(Here $\operatorname{Val}_{C \times D} (f(.,.))$ means the value of the matrix game on the polytope with extreme points the sets C respectively D for player 1 respectively player 2; for $(c,d) \in C \times D$ the payoff equals $f(c,d)$; A_s and B_s have the usual meaning and $E_s(k)$, $k = 1,2$, are the sets of extreme optimal actions for player k in the matrix game (4.2))

For each solution to (4.2)-(4.4), $g = (g_1,g_2,\ldots,g_z)$, is the same, equalling the average reward value of the stochastic game.
Equation (4.2) can be interpreted as the conservingness property in the sense that the players should take care, that they remain in their "good states" during the play.
Equation (4.3) for player a and (4.4) for player 2 reflect the equalizing property in the sense that within their good states the average rewards have to approach the value.
From a solution to (4.2)-(4.4) optimal stationary strategies can be constructed. Namely, let $\rho = (\rho(1),\rho(2),\ldots,\rho(z))$ be such that, for each $s \in S$, $\rho(s)$ is an optimal action for player 1 in the polyhedral game (4.3), then ρ is an optimal stationary strategy for player 1 in the average reward stochastic game. Likewise one can construct an optimal stationary strategy for player 2.

We already saw that optimal stationary strategies need not exist. One can wonder if for certain starting states one or both players can guarantee themselves the value for that starting state with the aid of stationary strategies. In Tijs and Vrieze (1986) it is shown that both players, in every game, have at least one state for which this is the case. It is still an open problem to characterize for a player his whole set of such "easy" states.

We finish this section by some remarks on games for which the value does not depend on the initial state. Already Bewley and Kohlberg (1978) showed that for games for which $\lim_{\beta\uparrow 1} (1-\beta)\, v_s^\beta$ does not depend on s, the value of the game exists and that both players possess optimal Markov strategies.
The following theorem can be found in Vrieze (1987)

Theorem 4.4

(a) For an average reward stochastic game the value is independent of the initial state if and only if, for some number $g \in \mathbb{R}$, for each $\varepsilon > 0$, the following set of equations has a solution for $v(\varepsilon) = (v_1(\varepsilon), v_2(\varepsilon), \ldots, v_z(\varepsilon))$:

$$|v_s(\varepsilon) + g - \underset{A_s \times B_s}{\mathrm{Val}} (r_s(.,.) + \Sigma_{t=1}^{z} p(t \mid s,.,.) v_t(\varepsilon))| \leqq \varepsilon, \text{ for each } s \in S. \tag{4.5}$$

(b) Both players have optimal stationary strategies in an average reward stochastic game with value independent of the initial state if and only if equation (4.5) has a solution for $\varepsilon = 0$ i.e. if and only if

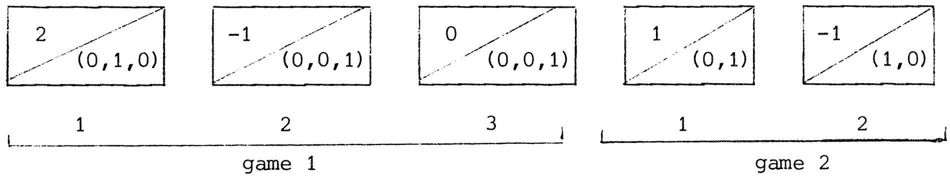
$$v_s + g = \underset{A_s \times B_s}{\mathrm{Val}} (r_s(.,.) + \Sigma_{t=1}^{z} p(t \mid s,.,.) v_t) \text{ for some } g \in \mathbb{R} \text{ and } v \in \mathbb{R}^z. \tag{4.6}$$

As well in part (a) as in part (b) of theorem 4.4 the value of the game is g for each starting state. In part (a) ε-optimal stationary strategies can be constructed by taking optimal actions in the matrix games in (4.5). In part (b) optimal stationary strategies result by taking optimal actions in the matrix games in (4.6).

5. TOTAL REWARD STOCHASTIC GAMES

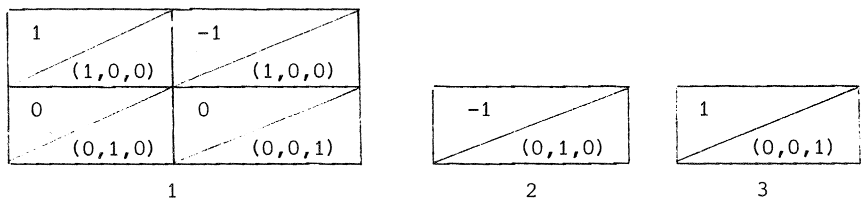
In section 3 we already mentioned that in fact Shapley considered total reward stochastic games under the restriction of stopping transitions.

In this section we apply the total reward criterion to stochastic games as defined in section 2. The motivations for looking at the total reward criterion lies in the fact that this can be seen as a sensitive criterion in addition to the average reward criterion.
For instance, consider the following examples :



For game 1, obviously the average reward value is (0,0,0). However player 1 would prefer to start in state 1 (getting total reward 1) and player 2 would prefer to start in state 2 (paying total reward - 1). Likewise in game 2 the average reward value is (0,0), but player 1 likes to start in state 1, thus owning half of the time one unit and half of the time zero units. And player 2 likes to start in state 2, being due half of the time minus one unit and half of the time 0 units.
For both games the average reward criterion does not discriminate between the states for the players, while the total reward criterion would do.

In general, total reward stochastic games need not have a value, as can be seen from the following example :



It can be easily be verified that for state 1 :

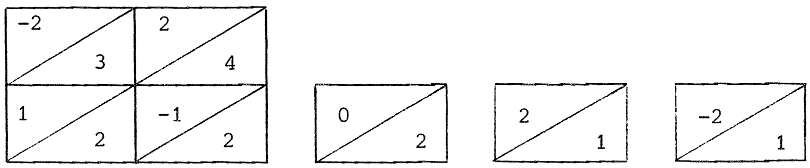
$$\sup_{\pi_1} \inf_{\pi_2} v^T(1,\pi_1,\pi_2) = -\infty \neq 0 = \inf_{\pi_2} \sup_{\pi_1} v^T(1,\pi_1,\pi_2).$$

Researchers in stochastic games will have recognized the above game as being the big match of Blackwell and Ferguson (1968).
It can be proved that in case one of the players has no optimal stationary strategy for an average reward stochastic game, then this game has no total reward value.
Therefore, we concentrate on games with the following property.

Property P1 : The stochastic game has average reward value (0,0,...,0) and both players possess optimal stationary strategies with respect to the average reward criterion.

This class of games is introduced in Thuijsman and Vrieze (1987). It is still an open question whether for this class of games the total reward value always exists. It can easily be shown that property P1 implies that both $\sup_{\pi_1} \inf_{\pi_2} v^T(s,\pi_1,\pi_2)$ and $\inf_{\pi_2} \sup_{\pi_1} v^T(s,\pi_1,\pi_2)$ are finite.

The following example, called the bad match, elaborated in Thuijsman and Vrieze (1987) shows that, in analyzing total reward games, similar problems as for average reward games are encountered.



1 2 3 4

For this game the total reward value equals (0,0,2,-2). Player 1 has no total reward optimal strategy, even more, player 1 has no ε-optimal (semi-)Markov strategy for the game starting in state 1 (or state 3 or state 4).

Observe the similarity between this game (the bad match) with the big match. In Thuijsman and Vrieze (1987) it is shown that total reward ε-optimal history dependent strategies for player 1 in the bad match can be constructed along the same lines as average reward ones in the big match.

In spite of this similarity, the analysis of Mertens and Neyman (1981) can not be implemented straightforward to total reward stochastic games, which is mainly due to the fact that even under property P1 streams of payoffs may occur for which the partial sums are not uniformly bounded.

Concerning the characterization of games with both players possessing total reward optimal stationary strategies, the following result can be mentioned : (Vrieze and Thuijsman (1986)).

Theorem 5.1

For a total reward stochastic game the value exists and both players have optimal stationary strategies if and only if the following set of functional equations has a solution $u = (u_1, u_2, \ldots, u_z)$, $w(1) = (w_1(1), w_2(1), \ldots, w_z(1))$, $w(2) = (w_1(2), w_2(2), \ldots, w_z(2)) \in \mathbb{R}^z$ and $\alpha \geq 0$:

$$u_s = \operatorname{Val}_{A_s \times B_s} (r_s(.,.) + \Sigma_{t=1}^{z} p(t \mid s,.,.)u_t), \text{ for each } s \in S \qquad (5.1)$$

$$w_s(1) + u_s = \operatorname{Val}_{E_s(1) \times B_s} (\alpha r_s(.,.) + \Sigma_{t=1}^{z} p(t \mid s,.,.)w_t(1)), \text{ for each } s \in S \qquad (5.2)$$

$$w_s(2) + u_s = \operatorname{Val}_{A_s \times E_s(2)} (\alpha r_s(.,.) + \Sigma_{t=1}^{z} p(t \mid s,.,.)w_t(2)), \text{ for each } s \in S \qquad (5.3)$$

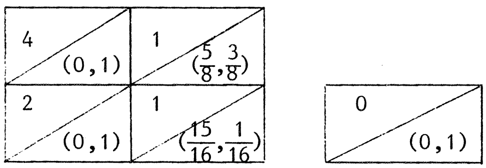
(Here $\operatorname{Val}_{C \times D}(f(.,.))$ has the same meaning as in theorem 4.3).

Observe the similarity of this theorem with theorem 4.3. Analogous to the average reward case, the u part of any solution to (5.1)-(5.3) is the same, being the total reward value. Also here optimal stationary strategies can be constructed from optimal actions in the polyhedral games (5.2) and (5.3). Notice further that equation (5.1) is equivalent to property P1 (cf part (b) of theorem 4.4). In case both players have total reward optimal stationary strategies it can be deduced from the limit discount equation that the total reward value equals $c(0) = (c_1(0), c_2(0), \ldots, c_z(0))$, i.e. the constant term of the Puiseux series solution to the limit discount equation. In this case $c(0)$ is also the leading term since under property P1 it holds that $c(1) = c(2) = \ldots = c(M) = 0$. In terms of the discount factor β this property can be stated as : total reward value = $\lim_{\beta \uparrow 1} v^\beta$. In Vrieze and Thuijsman (1986) it is conjectured that for games with property P1 the total reward value always equals $\lim_{\beta \uparrow 1} v^\beta$.

6. STRUCTURED STOCHASTIC GAMES

Since 1980 several subclasses of stochastic games are studied. Here subclasses of stochastic games are classes of stochastic games determined by conditions on the reward and/or payoff structure.

There are two reasons for analyzing structured games. First, practical situations seem to fit better into structural classes. Second, the value and optimal strategies can be computed more easy for structured games than in the general case.
In the general case neither the value nor optimal stationary strategies need to lie in the same ordered field as the data of the game. For instance the game with rational data :



has, for $\beta = 4/5$, discounted reward value $(\sqrt{8},0)$.

Stochastic games, for which the value and some pair of optimal stationary strategies, with respect to one of the three evaluation criteria, lie in the same ordered field as the data of the game, are said to have the orderfield property. Only for games having the orderfield property, one can expect to find a solution in a finite number of computation steps, resulting in an exact solution of the game. This observation gives further support to paying attention to structured games.

Successively we mention the classes studied so far and give in short some characteristics. We do not state properties with respect to the total reward criterion, since this criterion has only recently been proposed in the literature. However, most of the properties concerning the average reward criterion will also hold for the total reward criterion, when, in addition to the structure on the reward and transitions, property P1 (cf section 5) holds.

(a) One player controls transitions.

In this class, only one of the players controls the transitions. Say player 2, then $p(t \mid s,i_1,j) = p(t \mid s,i_2,j)$ for each $i_1,i_2 \in A_s$, each $j \in B_s$ and each $s,t \in S$. Hence we can denote the transitions by $p(t \mid s,j)$. The orderfield property for this class of games for as well the discounted as the average reward criterion, was first shown by Parthasarathy and

Raghavan (1981). For the discounted reward case they gave an LP algorithm. In Vrieze (1981), and independently in Hordijk and Kallenberg (1981), a constructive proof, using also an LP algorithm, for the average reward case can be found. Later on,this class of games is intensively studied by Filar (1984, 1987) especially as an application to the travelling inspector model.

(b) Transitions with switching control.

For this class of games in each state only one of the players governs the transitions. However, unlike the one player controls transition case, not in every state this has to be the same player.
This class of games was introduced by Filar (1981). For as well the discounted as the average reward case he proved the orderfield property. A constructive proof for the discounted version can be found in Vrieze (1987) and for the average version in Vrieze, e.a. (1983). In both cases the solution procedure consists of an iterative procedure of finite length, where at each iteration an LP problem has to be solved.

(c) SER-SIT games.

Separable reward and state independent transitions games are defined by the following structure : $r_s(i,j) = r_1(s) + r_2(i,j)$ and $p(t \mid s_1,i,j) = p(t \mid s_2,i,j)$ for each $s_1,s_2 \in S$. Hence we may write $p(t \mid i,j)$. As a consequence of the imposed structure the action sets of the players are the same for each state. These games are introduced in Parthasarathy, e.a. (1984). They showed that SER-SIT games have the orderfield property and that this class can be solved relatively.
For the discounted version this matrix game is
$[r_2(i,j) + \beta \, \Sigma_{t=1}^{z} \, p(t \mid i,j) \, r_2(t)]_{i=1}^{m} {}_{j=1}^{n}$.
For the average version this matrix game is the limit of the β-discounted one's for β tending to one :
$[r_2(i,j) + \Sigma_{t=1}^{z} \, p(t \mid i,j) \, r_2(t)]_{i=1}^{m} {}_{j=1}^{n}$.
For both criteria the both players have optimal myopic stationary strategies. Myopic means that the stationary strategy is even independent of the current state. A further result is that for the average case the value is

independent of the initial state. SER-SIT games are also partially studied by Sobel (1981).

(d) ARAT games.

Additive reward and additive transitions games are introduced by Raghavan, e.a. (1985). ARAT games are defined by $r_s(i,j) = r_{1s}(i) + r_{2s}(j)$ and $p(t \mid s,i,j) = p_1(t \mid s,i) + p_2(t \mid s,j)$. Hence both the rewards and the transitions are additive with respect to the both player.
They proved the following results. For as well the discounted as the average reward criterion both players possess optimal stationary pure strategies. This property immediately implies the orderfield property. Further, both players have uniformly discount optimal stationary pure strategies. These results follow straightforward from Shapley's equations (cf (3.2)), since the matrix game (3.1) can be decomposed in a term depending on i and a term depending on j for ARAT games.

(e) One player controls rewards for a game with two states.

This class of games is introduced by Vrieze, e.a. (1986).
It is defined by : restriction to two states and $r_s(i,j) = \tilde{r}_s(i)$, i.e. the rewards only depend on the action of player 1. Also for these class of games the orderfield turns out to hold for the discounted case. For the average reward criterion this is an open question.

We conclude this section by the remark that the time has come to characterize the subclass of games having the orderfield property. Two approaches look promising at least for the discounted reward criterion.
A first one is established in the paper by Vrieze, e.a. (1986). To each set of stationary pure strategies of a player they add a set of stationary strategies of the other player in the following way :
Let Q be a set of stationary pure strategies for say player 1 and S(Q) the stationary strategies of player 2 added to Q, then $\sigma \in S(Q)$ if and only if each $\rho \in Q$ is a best answer to σ.
Vrieze, e.a. (1986) showed that , in their case, S(Q) is either void or a union of a finite number of disjoint polytopes with rational extremes (when the data is rational). In general this quality is enough for proving the

orderfield property. And as such this idea can be used for characterizing the orderfield property in more generally settings.

A second approach can be found in Sinha (1986). He combined SER-SIT games and switching control games. For the discounted case he exploited a value iteration method based on Shapley's equations. In each step three connected LP problems have to be solved. In a finite number of steps the solution is reached. His technique of proof relies on the fact that the solution of the discounted reward game corresponds to an extreme point of a suitable chosen system of linear inequalities. Thus corresponding to a certain base of this system. In each iteration step this system comes back together with some base. Since each of his iterations approaches the value better and since there are a finite number of different base, Sinha was able to prove that his procedure stops after a finite number of steps. There are indications that this method can be extended to generally proving whether some subclass has the orderfield property (in the discounted case) or not.

7. ALGORITHMS FOR STOCHASTIC GAMES

In this final section we give a short review on algorithm for discounted and average reward stochastic games. Always the question is, how to compute the value of the game together with a pair of stationary strategies (when existing). For solution methods for special subclasses we refer to section 6.

(a) Algorithms for discounted reward stochastic games.

In the first place we mention the algorithm, which, in a natural way arises from Shapley's proof of the existence of the value, namely (cf (3.1) and (3.2)) :

1. choose $v_0 = (v_0(1), v_0(2), \ldots, v_0(z))$ arbitrary
2. let, for $\tau = 1,2,\ldots$, : $v_\tau(s) := \mathrm{Val}(M_s^\beta(v_{\tau-1}))$, each $s \in S$.

This value iteration method approaches the value of the game exponentially fast, while at each iteration suboptimal stationary strategies can be deduced from the matrix games $M_s^\beta(v_{\tau-1})$.

A second algorithm, proposed by Hoffman and Karp (1966), can be named as value oriented policy iteration. It runs as follows :

1. choose $v_0 = (v_0(1), v_0(2), \ldots, v_0(z))$ arbitrary.
2. let $\tau = 0,1,2,\ldots$, $\sigma^\tau = (\sigma^\tau(1), \sigma^\tau(2), \ldots, \sigma^\tau(z))$ be such that $\sigma^\tau(s)$ is an optimal action for player 2 in $M_s^\beta(v_\tau)$.
3. solve for player 1 the Markov decision problem, which results when player 2 fixes σ^τ and let $v_{\tau+1}$ be the optimal value; repeat from 2.

A third algorithm we mention is an extension of the Brown-Robinson scheme for matrix games to stochastic games (Vrieze and Tijs (1982)). First they showed that the scheme can be applied at a converging sequence of matrix games. Next the contraction property of Shapley's value operator enables them to proof the convergence of the Brown-Robinson scheme when applied to discounted stochastic games.
The convergence rate is low (the same as in the case of matrix games), however at each iteration only simple calculations have to be done.

More about algorithms, especially viewed in a mathematical programming context (cf also Vrieze (1987)), can be found in Schultz (1987).

Several facts about convergence rates for successive approximation schemes and value oriented policy iteration schemes can be found in Van der Wal (1981).

(b) Algorithms for average reward stochastic games.

For average reward stochastic games there are still many open problems. The existing algorithms only solve special classes. Surely, by the result of Bewley and Kohlberg (1976) (cf section 4), the average value, g, can be approached by computing the discounted value, v^β, and letting β tend to 1 ($g = \lim_{\beta \uparrow 1}(1-\beta)v^\beta$). However there are no clear rules available to estimate the convergence rate. A further difficulty is that the players need not possess optimal stationary strategies.

We mention two algorithm.
The first one is the application of the Hoffman and Karp schemes to average reward games. However restricted to the class of irreducible stochastic games, i.e. games for which for each pair of stationary pure strategies the corresponding stochastic matrix (cf 2.3) has a single ergodic

class and no transient states.
They showed that their scheme converges to a solution of the following set of equations (in $g \in \mathbb{R}$ and $v = (v_1,\dots,v_z) \in \mathbb{R}^z$)

$$g + v_s = \operatorname*{Val}_{A_s \times B_s} (r_s(.,.) + \Sigma_{t=1}^{z} p(t \mid s,.,.)v_t), \text{ each } s \in S. \tag{7.1}$$

By theorem 4.4, part (b) it follows at once that a solution to these equations is equivalent to a solution of the average reward stochastic game.

A second algorithm is due to Federgruen (1984) and can be applied to games for which (1) both players have optimal stationary strategies and (2a) the average value is independent of the initial states or (2b) the stochastic game is irreducible. Federgruen's scheme is an extension of the modified value-iteration method of Hordijk and Tijms (1975) for Markov decision problems to stochastic games. The idea is to choose a suitable sequence of discount factors tending to 1, obeying certain desired properties. Next at each step a (discounted) value iteration step is carried out, resulting in a scheme which converges to a solution of the set of equations (7.1). Related algorithm for the same classes can be found in Van der Wal (1981).

Since playing stationary is preferable to playing nonstationary it would be nice if there was some algorithm yielding $\sup_{\sigma} \inf_{\pi_2} g(\rho,\pi_2)$ for player 1 and $\inf_{\sigma} \sup_{\pi_1} g(\pi_1,\sigma)$ for player 2. Observe that these quantities are the bounds for the respective players that can be reached by playing stationary. Only in case both players possess ε-optimal stationary strategies, these bounds are the same, equalling the average value of the game. No algorithm for this problem is at hand. An adaption of the above mentioned algorithm of Federgruen, using extensions of the characterization in theorem 4.3, looks a promising candidate. However no convergence proof is available yet.

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CHAPTER V

NON-ZEROSUM STOCHASTIC GAMES

by Frank Thuijsman *)

1. INTRODUCTION

The theory of stochastic games was born in 1953 by the appearance of a paper titled "Stochastic games", written by L.S. Shapley (1953). This paper has inspired many people to examine stochastic games in order to get a better understanding of this complex and challenging kind of games, in which a link is made between Markov decision problems and finite non-cooperative games. In fact Markov decision problems as well as bimatrix games and repeated games can be seen as special kinds of stochastic games. Stochastic games are also connected with stochastic processes, specially with Markov chains, and therefore these games are also known as Markov games.

This chapter is arranged as follows. In section 2 we give the formal definitions that are of importance and we take a first look at an example which will be worked out in section 5. In section 3 we mention some results on zerosum stochastic games, which are necessary for a better understanding of non-zerosum stochastic games. In section 4 we give a review on non-zerosum stochastic games. In section 5 we work out an example. We end this chapter with some concluding remarks in section 6. For more information about zerosum stochastic games we refer to the chapter by Vrieze.

Before turning to the formal definitions we give two ways of looking at a stochastic game. Those who are familiar with Markov decision problems may look at a two person stochastic game as a Markov decision problem for which in each state and on each stage, instead of one person, there are two people who simultaneous and independent of each other, each have to choose an action. Now the payoffs to the players as well as the transition

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to the next state are governed by the current state and the pair of actions chosen. Since the players in general do not have the same interest, normally do not know what the other is going to do and are not allowed to make binding agreements, we have a non-cooperative game situation.

Those familiar with bimatrix games could think of a two person stochastic game as a finite collection of bimatrix games to be played, every day one and never stopping, where the actions chosen on a certain day in some bimatrix not only determine a payoff to each player but also determine a probability vector over the set of bimatrices, according to which the play moves to a new bimatrix game to be played next day. So instead of playing one bimatrix game one time and keeping account of the direct payoff only, in the stochastic game the players have to look at the possibilities of current payoffs and future prospects at the same time. It should be clear that each player's objective is to get as rich as possible without making binding agreements, but perhaps by making use of threats (like in repeated games).

In this chapter we restrict our attention to two person non-zerosum stochastic games with finite state and action spaces although several of the results mentioned also hold for more general stochastic games. We also assume the players to have complete information, i.e. at each stage both players know the current state as well as the sequence of states and actions that have been visited, respectively chosen, up to that stage. The players also know for all states and pairs of possible actions the corresponding direct payoffs and transition vector.

2. GETTING INTO THE STOCHASTIC GAME MODEL

2.1.1. *Stochastic games; the situation*

A two person stochastic game is a set of matrices $\{M_s : s \in S\}$, where $S = \{1,2,\dots,z\}$. Matrix M_s has size $m(s) \times n(s)$ and for $i \in A_s = \{1,2,\dots,m(s)\}$ and $j \in B_s = \{1,2,\dots,n(s)\}$ entry (i,j) of M_s is of the form :

$r_1(s,i,j), r_2(s,i,j)$ ╱ $p(s,i,j)$

where $r_1(s,i,j)$, $r_2(s,i,j) \in \mathbb{R}$ and $p(s,i,j) = (p(1 \mid s,i,j), p(2 \mid s,i,j), \ldots, p(z \mid s,i,j)) \in P(S) = \{x \in \mathbb{R}^z : x \geq 0 \text{ and } \sum_{t=1}^{z} x_t = 1\}$.
The elements of S are called states.
For each $s \in S$ the elements of A_s, respectively B_s, are called the pure actions of player I, respectively player II, in state s.
For each $s \in S$, $i \in A_s, j \in B_s, r_1(s,i,j)$ and $r_2(s,i,j)$ are the payoffs to players I and II respectively, if in state s player I chooses i and player II chooses j.
For all $s,t \in S, i \in A_s, j \in B_s, p(t \mid s,i,j)$ is the probability that the play moves from state s to state t if in state s player I chooses i and player II chooses j.
Unless mentioned otherwise we will deal with $\mathbb{N} = \{1,2,3,\ldots\}$ as set of stages at which the players have to choose actions.
In case for all s,i and j $r_2(s,i,j) = -r_1(s,i,j)$ the game is called a zerosum stochastic game.

2.1.2. *Stochastic games; the rules*

A stochastic game, as just defined, is played in the following way. Each of the states can be considered as starting state of the play at stage 1. At each stage $n \in \mathbb{N}$ the play is in exactly one of the states; suppose it is in state s at stage n. Then, simultaneous and independent of each other player I has to choose an action out of A_s and player II has to choose an action out of B_s. Suppose player I chooses i and player II chooses j. Then each of them informs the impartial referee about his choice. The referee anounces the pair of choices and orders the bank to pay $r_1(s,i,j)$ to player I and $r_2(s,i,j)$ to player II. Finally for this stage, the referee carries out a chance experiment of which the outcome will be t with probability $p(t \mid s,i,j)$, for all $t \in S$. The outcome of this experiment determines the state at which the play will be at stage n+1, the next stage. As such the play goes on indefinitely, moving from state to state.

To get familiar with the stochastic game model, we now take a brief look at an example.

2.2. *A bankrobbery game*

Think of a small town with 10 banks, one policeman and one thief. Every evening the thief can decide either to get to bed or to go out and rob one of the banks. Similarly the policeman can decide to go to bed or to go out to watch over one of the ten banks. The policeman has a fixed salary of 2 daily, but going out to watch over a bank takes effort costing him 1. On the other hand, staying home will cost him 1 in case there is a bankrobbery that night, for then he is to blame for not doing his duty well. We even assume that each time the latter occurs the policeman has probability 1/10 of getting fired, in which case his daily income reduces to 0, and the thief would then receive 10 each day. The thief earns 10 in case he robs a bank and he earns 0 if he stays at home. In case the thief decides to rob a bank he has probability 1/10 of getting caught and sent to prison if the policeman is on guard. In prison the thief's daily income is -1, while the policeman would simply get his salary 2 daily. Once in prison, or once fired it will stay that way forever.

We can model this situation as a stochastic game with 3 states :

2,0 (1,0,0)	1,10 (9/10,0,1/10)
1,0 (1,0,0)	1,10 (9/10,1/10,0)

1

2,-1 (0,1,0)

2

0,10 (0,0,1)

3

Player I, the row player, is the policeman and player II, the column player, is the thief. State 2 is the prison and in state 3 player I has been fired. Of course, the interesting starting state is state 1. The question is what the policeman and the thief could do, without making binding agreements, to "maximize" their individual incomes. An answer will be given in section 5.

2.3. *Strategies*

A player, facing a stochastic game, should make a strategy that tells him for all possible situations, i.e. states, stages and sequences of past events, what to do. We allow the players to randomize over their pure actions, just like in bimatrix games. This means that each player is allowed to carry out a chance experiment to decide which pure action to choose. So in fact in state $s \in S$ player I can choose a mixed action from $P(A_s) = \{x \in \mathbb{R}^{m(s)} : x \geq 0, \sum_{i=1}^{m(s)} x_i = 1\}$, which should be interpreted as player I choosing pure action i with probability x_i. We identify the mixed action e_i = choosing i with probability 1, with the pure action i. The players never get to know the mixed actions that are used in the decision making by their opponent. The players are only informed about the pure actions that have actually been chosen, and those can be the "results" of using some unknown mixed actions. So a strategy should tell a player exactly what mixed action he should use for any given situation. This brings us to the following definition :

A strategy for player I (resp. II) is a function that assigns to each triple (s,n,h_n), with $s \in S$, $n \in \mathbb{N}$ and $h_n = (s_1,i_1,j_1,s_2,i_2,j_2,\ldots,s_{n-1},i_{n-1},j_{n-1})$, an element of $P(A_s)$ (resp. $P(B_s)$). Here h_n is the history of the play at stage n : at stage $k \in \{1,2,\ldots,n-1\}$ the play was in state $s_k \in S$, player I chose $i_k \in A_{s_k}$ and player II chose $j_k \in B_{s_k}$. Strategies will be denoted π_1 and π_2 for players I and II respectively. A strategy π is called :

- a stationary strategy if for all (s,n,h_n) : $\pi(s,n,h_n) = \pi(s)$
- a Markov strategy if for all (s,n,h_n) : $\pi(s,n,h_n) = \pi(s,n)$
- a behaviour strategy if π depends on h_n.

Stationary strategies are denoted ρ for player I and σ for player II.

2.4. *Evaluation criteria*

Given a starting state s and strategies π_1 and π_2 for players I and II respectively, one can iteratively compute the expected payoff $E_{s\pi_1\pi_2}(R_k(n))$ to player $k \in \{1,2\}$ at stage $n \in \mathbb{N}$. So for player $k \in \{1,2\}$ we have a stream of expected payoffs $(E_{s\pi_1\pi_2}(R_k(1)), E_{s\pi_1\pi_2}(R_k(2)),\ldots)$. As we mentioned before, each player wants to earn as much as possible. What does this mean for infinite sequences of expected payoffs ? It is obvious that the players need some evaluation criterion to relate each infinite stream of expected payoffs with a single expected reward, the worth of the stream. We will only consider stochastic games in which both players evaluate streams of expected payoffs by using the same evaluation criterion. There are several ways to evaluate a stream of expected payoffs. Two evaluation criteria have been studied extensively : the β-discount criterion and the average criterion. A third one is the total reward criterion.

2.4.1. β-*Discount stochastic games*

A β-discount stochastic game is a stochastic game in which both players use the β-discount criterion. Here $\beta \in (0,1)$ and player $k \in \{1,2\}$ evaluates $(E_{s\pi_1\pi_2}(R_k(1)), E_{s\pi_1\pi_2}(R_k(2)),\ldots)$ as $v_k^\beta(s,\pi_1,\pi_2) = \sum_{n=1}^{\infty}\beta^{n-1}E_{s\pi_1\pi_2}(R_k(n))$. $v_k^\beta(s,\pi_1,\pi_2)$ is the β-discounted income (reward) of player k if the play starts in state s, player I uses strategy π_1 and player II uses strategy π_2. We also write $v_k^\beta(\pi_1,\pi_2) = (v_k^\beta(1,\pi_1,\pi_2),\ldots, v_k^\beta(z,\pi_1,\pi_2))$.

The idea in β-discount stochastic games is, that the players discount future payoffs by a factor $\beta \in (0,1)$ which corresponds with a rate of interest $(1-\beta)/\beta$. A payoff x at the n-th stage is worth the same as a payoff $\beta^{n-1}x$ at the first stage, since $\beta^{n-1}x$ at stage 1 grows, under interest rate $(1-\beta)/\beta$ per stage, to x at stage n. It is clear that the β-discounted incomes are mainly determined by the expected payoffs in the beginning of the play since all expected payoffs are bounded and β^{n-1} tends to zero as n increases.

In non-zerosum stochastic games, the players trying to maximize their income could look for stable pairs of strategies, called equilibria :

For $\varepsilon > 0$ a β-discount ε-equilibrium is a pair of strategies (π_1^*,π_2^*) such that for all π_1,π_2 and $s \in S$:

$v_1^\beta(s,\pi_1^*,\pi_2^*) \geqq v_1^\beta(s,\pi_1,\pi_2^*)-\varepsilon$ and

$v_2^\beta(s,\pi_1^*,\pi_2^*) \geqq v_2^\beta(s,\pi_1^*,\pi_2)-\varepsilon$.

If one can take $\varepsilon = 0$, then (π_1^*,π_2^*) is called an equilibrium.

For an ε-equilibrium (π_1^*,π_2^*), π_1^* is an ε-best answer for player I against

π_2^*, and π_2^* is an ε-best answer for player II against π_1^*. So once the players would have agreed somehow to play (π_1^*,π_2^*) neither player could gain more than ε by one-sidedly starting to play another strategy. Thus there is some stability in (π_1^*,π_2^*). The above definition of ε-equilibrium is due to Nash (1951), thus those equilibria are also called Nash-equilibria.

A β-discount Nash equilibrium payoff, β-Nep, is a pair $(x,y) \in \mathbb{R}^z \times \mathbb{R}^z$ for which there exists a sequence $(\pi_1^{\varepsilon(n)},\pi_2^{\varepsilon(n)})_{n \in \mathbb{N}}$ such that :

i) $\lim_{n\to\infty} \varepsilon(n) = 0$

ii) $(\pi_1^{\varepsilon(n)},\pi_2^{\varepsilon(n)})$ is a β-discount ε_n-equilibrium

iii) $\lim_{n\to\infty} (v_1^\beta(\pi_1^{\varepsilon(n)},\pi_2^{\varepsilon(n)}),\ v_2^\beta(\pi_1^{\varepsilon(n)},\pi_2^{\varepsilon(n)})) = (x,y)$.

So if (π_1^*,π_2^*) is a β-discount equilibrium then $(v_1^\beta(\pi_1^*,\pi_2^*),\ v_2^\beta(\pi_1^*,\pi_2^*))$ is the corresponding β-Nep. Different equilibria can correspond with different Neps.

In zerosum stochastic games the players have strictly opposite interests. Then we can talk of the value of the game and of ε-optimal strategies : A β-discount zerosum stochastic game has a (β-discount) value $v^\beta = (v^\beta(1), v^\beta(2),\ldots,v^\beta(z))$ if for all $\varepsilon > 0$ there exists π_1^* and π_2^* such that for all π_1,π_2 and $s \in S$:

$$v_1^\beta(s,\pi_1^*,\pi_2) + \varepsilon \geqq v_1^\beta(s) \geqq v_1^\beta(s,\pi_1,\pi_2^*) - \varepsilon.$$

π_k^* is called a β-discount ε-optimal strategy for player $k \in \{1,2\}$. π_1^*, resp. π_2^*, is called a β-discount optimal strategy for player I, resp. II, if one can take $\varepsilon = 0$ for the left-side, resp. right-side, inequality.

2.4.2. *Average stochastic games*

An average stochastic game is a stochastic game in which both players use the average criterion. This means that player $k \in \{1,2\}$ evaluates the stream of expected payoffs $(E_{s\pi_1\pi_2}(R_k(1)),\ E_{s\pi_1\pi_2}(R_k(2)),\ldots)$ as

$g_k(s,\pi_1,\pi_2) = \liminf_{T\to\infty} \frac{1}{T} \sum_{n=1}^{T} E_{s\pi_1\pi_2}(R_k(n))$. $g_k(s,\pi_1,\pi_2)$ is the average income of player $k \in \{1,2\}$ for the play starting in state s and the players using π_1 and π_2 respectively. Let $g_k(\pi_1,\pi_2) = (g_k(1,\pi_1,\pi_2),\ldots,g_k(z,\pi_1,\pi_2))$.

Average stochastic games were introduced by Gillette (1957). Whereas in β-discount stochastic games the early expected payoffs have the main impact on the β-discounted incomes, in average stochastic games the players are mainly concerned about the far-future expected payoffs.

Average (ε-) equilibria and average Neps, g-Neps, are defined analogous to β-discount ones. Also for zerosum average stochastic games the average value g and average (ε-) optimal strategies are defined analogous to those in the β-discount stochastic game (replace v_k^β by g_k and β-discount by average in the definitions in 2.4.1).

2.4.3. *Total reward stochastic games*

A third criterion makes sense for zerosum stochastic games which have average value 0 ($\in \mathbb{R}^z$) and for which both players possess average optimal stationary strategies. This criterion is the total reward criterion, introduced by Thuijsman and Vrieze (1987). For the total reward (income) the expected payoffs in the beginning of a play are equally important as far future expected payoffs. For more about this criterion we refer to the chapter by Vrieze. Here we will only give the definition :

A total reward stochastic game is a zerosum stochastic game with average value 0 and each player possessing an optimal stationary strategy. For strategies π_1 and π_2 and starting state s player I's total income is given by :

$$v^T(s,\pi_1,\pi_2) = \liminf_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \sum_{n=1}^{t} E_{s\pi_1\pi_2}(R_1(n)).$$

Note that in case $\sum_{n=1}^{\infty} E_{s\pi_1\pi_2}(R_1(n))$ exists, it is equal to $v^T(s,\pi_1,\pi_2)$. For total reward stochastic games a value v^T and ε-optimal strategies are defined analogous to those for β-discount stochastic games.

2.5. *The I-zerosum stochastic game and the II-zerosum stochastic game*

Any non-zerosum stochastic game is related with two zerosum stochastic games by looking at the payoffs to one of the players only and assuming that they have to be paid by the other player. If we only look at the payoffs to player I we get what we call the I-zerosum stochastic game. In case this game has a value it will be denoted v^β or g, corresponding to the criterion being used. If we only look at the payoffs to player II we get the II-zerosum stochastic game of which the value, to player II, will be denoted $\bar{v}^\beta$ or $\bar{g}$, depending on the criterion. Note that usually a zerosum stochastic game is considered from the point of view of player I for whom the payoffs are given. Hence in the definition of the value of the II-zerosum game the roles of the players are interchanged in comparison with the definition of value in 2.4.1.

2.6. *Some special stochastic games*

A bimatrix game is a two-person stochastic game with just one state and only one stage to go. A matrix game is a zerosum bimatrix game, in which case mostly the payoffs to player I only are given.

A repeated game is a two-person stochastic game with just one state. So it is a bimatrix game that is played over and over again.

A repeated game with absorbing states is a stochastic game in which all but one states are absorbing, i.e. once such a state is reached there is no way of leaving it again. Such a game can be seen as repeatedly playing a bimatrix game of which some entries are absorbing.

2.7. *The ordered field property*

As we will see in section 3 for every zerosum stochastic game the β-discount value and the average value exist, and so do β-discount optimal stationary strategies. For non-zerosum stochastic games β-discount stationary equilibria always exist. For the average criterion the existence of ε-equilibria in general is still an open question though we have good indications that this question will be answered positively in the near future (cf. section 4). However, to find a solution to a stochastic game is often a hard struggle. Therefore one often restricts the attention to special classes of stochastic games, hoping that they are more easy to solve than general stochastic games. Suppose all data determining a stochastic game are rational, then it would be interesting to know if a solution, i.e. value, (ε-)optimal strategies, (ε-)equilibrium, can be found of which all components are rational as well. If one would have this "ordered field property" one could hope that a finite algorithm exists to solve the game. Even for very simple stochastic games the ordered field property need not hold. However in section 4.6 several special classes are given which do have the ordered field property.

3. ABOUT ZEROSUM STOCHASTIC GAMES

For a good understanding of non-zerosum stochastic games it is necessary to know some important facts of zerosum stochastic games. Therefore we now take a brief look at them, starting of with matrix games. For more detailed information we refer to the chapter by Vrieze.

3.1. *Von Neumann's minimax theorem*

Von Neumann (1928) showed that for every $m \times n$ matrix $M = [a_{ij}]_{i=1}^{m}{}_{,j=1}^{n}$ it holds that

$$\max_{p \in S^m} \min_{q \in S^n} p\,Mq^t = \min_{q \in S^n} \max_{p \in S^m} p\,Mq^t$$

where for $k = m,n$ $S^k = \{x \in \mathbb{R}^k;\ x \geq 0,\ \sum_{i=1}^{k} x_i = 1\}$.
This can be interpreted as follows :
every matrix game M has a value and each player has an optimal mixed action, i.e. there exist $v \in \mathbb{R}$, $p^* \in S^m$ and $q^* \in S^n$ such that for all p and q : $p^*Mq^t \geq v \geq pMq^{*t}$; v is unique and denoted val[M] or val$[a_{ij}]$.
This theorem is very important in the theory of stochastic games, as will soon turn out.

3.2. *Shapley's stochastic games*

In the fundamental paper on stochastic games Shapley (1953) considers zerosum stochastic games for which for every pair of actions in any state there is a strictly positive probability that the play stops. Such games are considered as total reward stochastic games. Stopping can be seen as moving to an absorbing state with payoff 0. Let $S = \{1,2,\ldots,z,z+1\}$ where z+1 corresponds to stopping. For $x \in \mathbb{R}^z$ define $Tx \in \mathbb{R}^z$ by

$$(Tx)_s = \text{val}[r(s,i,j) + \sum_{t=1}^{z+1} p(t \mid s,i,j)x_t], \text{ in which } x_{t+1} = 0.$$

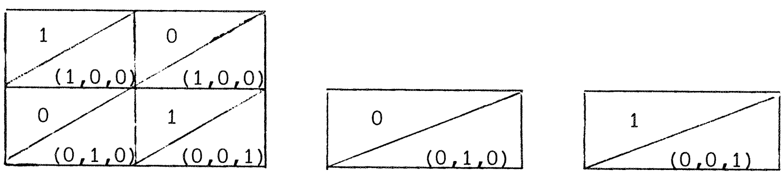
Shapley (1953) shows that T is a continuous contraction mapping and T has a unique fixed point $v \in \mathbb{R}^z$. Shapley proofs that v is the total reward value of the stochastic game and that optimal stationary strategies correspond 1 to 1 to optimal mixed actions in the matrix games $M_s(v) = [r(s,i,j) + \sum_{t=1}^{z+1} p(t \mid s,i,j)v_t]$ for $s \in \{1,2,\dots,z\}$. So a solution to the stochastic game can be found by solving Tx = x, the "Shapley-equation".

3.3. β-*Discount stochastic games*

From Shapley's work one can directly conclude that any β-discount zero-sum stochastic game has a value v^β. This follows because discounting by a factor β can be seen as stopping with probability 1-β from every entry, and looking at the stochastic game as a total reward stochastic game. Hence for every β-discount stochastic game the unique solution of the system of z equalities $x_s = \text{val}[M_s^\beta(x)]$, $s \in S$, with $x \in \mathbb{R}^z$ and $M_s^\beta(x) = [r(s,i,j) + \beta \sum_{t=1}^{z} p(t \mid s,i,j)x_t]$, is v^β and a stationary strategy ρ for player I, resp. σ for player II, is β-discount optimal if and only if for each s ρ_s for player I, resp. σ_s for player II, is an optimal mixed action in the matrix game $M_s^\beta(v^\beta)$.

3.4. *Average stochastic games*

Gillette (1957) was the first to study average stochastic games. He gave the following example of which it was uncertain a long time whether it had a value. Later it became known as the big match.



1 2 3

The payoffs are those to player I, to be paid by player II. The states 2 and 3 are absorbing, so state 1 is the interesting starting state. Note

that as long as player I chooses the first row, the play will remain in state 1; as soon as player I chooses the second row, the play moves to state 2 or state 3 depending on player II's choice.
It is not hard to show that if player I restricts to Markov strategies then $\sup_{\pi_1^M} \inf_{\pi_2} g(1,\pi_1^M,\pi_2) = 0$ and $\inf_{\pi_2} \sup_{\pi_1^M} g(1,\pi_1^M,\pi_2) = \frac{1}{2}$, even if player II restricts to Markov strategies as well.

For a long time it was unknown whether the value existed for this game. Blackwell and Ferguson (1968) solved the problem by showing that the average value for starting state 1 is $g_1 = 1/2$. They gave ε-optimal behaviour strategies for player I. In their paper one can also find that no average optimal strategy exists for this game. It was the start of solving the problem whether or not the average value exists for any zerosum stochastic game.

Kohlberg (1974) extended the work of Blackwell and Ferguson (1968) by describing how any repeated game with absorbing states can be solved. So any repeated game with absorbing states has a value and each player has ε-optimal (behaviour) strategies.

Then Bewley and Kohlberg (1978) noted that in case a player has a stationary strategy ρ^* which is β-discount optimal for all β close to 1, then ρ^* is also average optimal and the average value g exists and is related to the β-discount values by $g = \lim_{\beta\uparrow 1} (1-\beta)\, v^\beta$.

Finally Monash (1979) and Mertens and Neyman (1981) independently showed that for every zerosum stochastic game the average value exists and $g = \lim_{\beta\uparrow 1} (1-\beta)\, v^\beta$ always. They obtained this result by showing that average ε-optimal strategies can consist of playing the proper β-discount optimal strategies while β varies and tends to 1.

3.5. *Remark*

Notice that a limit of β-discount optimal strategies, β going to 1, certainly need not be optimal. Look for instance at the big match : $v_2^\beta = 0$, $v_3^\beta = 1/(1-\beta)$ and v_1^β can be derived by solving the Shapley equation :

$v_1^\beta = \text{val}\begin{bmatrix} 1+\beta v_1^\beta & \beta v_1^\beta \\ 0 & 1/(1-\beta) \end{bmatrix}$ which gives us $v_1^\beta = 1/(2-2\beta)$. The unique β-discount optimal strategy for player I is $\rho^\beta = (1/(2-\beta), (1-\beta)/(2-\beta))$ and for player II it is $\sigma^\beta = (\frac{1}{2},\frac{1}{2})$. Then $\rho^1 = \lim_{\beta\uparrow 1} \rho^\beta = (1,0)$. Let $\sigma_1 = (1,0)$ and $\sigma_2 = (0,1)$. Then $g(1,\rho^\beta,\sigma_1) = 0$ for all $\beta < 1$ whereas $g(1,\rho^1,\sigma_1) = 1$, and $g(1,\rho^\beta,\sigma_2) = 1$ for all $\beta < 1$ whereas $g(1,\rho^1,\sigma_2) = 0$. The average value of the big match is $g = \lim_{\beta\uparrow 1} (1-\beta)\, v^\beta = (\frac{1}{2},0,1)$. Hence neither is ρ^1 average optimal nor is ρ^β average ε-optimal for β close to 1. As we see there is a discontinuity in the average income of player I playing ρ^β against several fixed pure stationary strategies, as β goes to 1. This discontinuity corresponds with a discontinuity in the underlying Markov chains : for every ρ^β state 1 is transient whereas for ρ^1 state 1 is recurrent. This discontinuity makes average stochastic games difficult to play; sophisticated behaviour strategies can be indispensable to play average ε-optimal. However there are several special classes of stochastic games in which average optimal stationary strategies exist. (See the chapter by Vrieze).

4. ABOUT NON-ZEROSUM STOCHASTIC GAMES

4.1. *Introduction*

In non-zerosum stochastic games the players do not necessarily have strictly opposite interests. Player I could think that player II will do his best to minimize the income of player I, regardless whether this is also good for player II. This would make player I choose an (ε-)optimal strategy in the I-zerosum stochastic game. However, player II has other interests so perhaps player I could do something better, but what ? Of course player II faces a similar problem.

The idea in non-zerosum games is that the players, looking for some stability, would like to play an (ε-)equilibrium : a pair of strategies (π_1,π_2) such that π_1 is an ε-best answer for player I against π_2 and, at the same time, π_2 is an ε-best answer for player II against π_1. Hence, once they have come to play an ε-equilibrium neither player I nor player II would have a reason to start doing something else one-sidedly. How the players should actually get to play an (ε-)equilibrium is a cooperative problem which we shall not deal with. One should note that different equilibria in the same game may correspond with different incomes for the players. Hence the players in general have a different preference for the equilibrium they had liked to be played.

When playing a non-zerosum stochastic game, each player can always be sure that his income will be at least the value ($-\varepsilon$) of his, i.e. I- or II-, zerosum game. Thus in case (π_1,π_2) is an (ε-)equilibrium then for each player the corresponding income is at least the value ($-\varepsilon$) of his zerosum game. This follows directly from the definitions of value and ε-equilibrium. Suppose for instance that we are dealing with an average stochastic game in which for each $\delta > 0$ π_1^δ is a δ-optimal strategy for player I in the I-zerosum game and suppose that (π_1^*,π_2^*) is an average ε-equilibrium. Then $g_1(\pi_1^*,\pi_2^*) \geqq g_1(\pi_1^\delta,\pi_2^*) - \varepsilon \geqq g_1 - \delta - \varepsilon$ for all $\delta > 0$. Hence $g_1(\pi_1^*,\pi_2^*) \geqq g_1 - \varepsilon$.

Do ε-equilibria exist in any two-person stochastic game ? This question has been answered in the affirmative for many classes. We are now going to take a look at them.

4.2. *Bimatrix games*

For every bimatrix game there exists at least one equilibrium. In fact Nash (1951) prove that every finite non-cooperative n-person game has an equilibrium. We sketch the proof of Nash for an $m \times n$ bimatrix game : Let S^m, resp. S^n, be the set of mixed actions for player I, resp. player II, with pure actions $e_1,e_2,\ldots,e_m$ and $f_1,f_2,\ldots,f_n$ respectively. For a pair of (mixed) actions (ρ,σ) let $r_k(\rho,\sigma)$ be the expected payoff to player $k \in \{1,2\}$. For $i = 1,2,\ldots,m$ define $s_i(\rho,\sigma) = \max\{0,r_1(e_i,\sigma) - r_1(\rho,\sigma)\}$ and for $j = 1,2,\ldots,n$ define $t_j(\rho,\sigma) = \max\{0,r_2(\rho,f_j) - r_2(\rho,\sigma)\}$. Finally define

$$T(\rho,\sigma) = \left(\frac{\rho + s(\rho,\sigma)}{1 + \sum_{i=1}^m s_i(\rho,\sigma)}, \frac{\sigma + t(\rho,\sigma)}{1 + \sum_{j=1}^n t_j(\rho,\sigma)}\right).$$

One can show that T is a continuous function from the compact and convex set $S^m \times S^n$ into itself. Hence one can apply the Brouwer fixed point theorem (cf. Kakutani (1941)) which gives us the existence of a pair (ρ^*,σ^*) for which $T(\rho^*,\sigma^*) = (\rho^*,\sigma^*)$. It is easy to verify that a pair (ρ,σ) is an equilibrium if and only if $T(\rho,\sigma) = (\rho,\sigma)$. So, the Brouwer fixed point theorem immediately gives us the existence of at least one equilibrium.

For methods to actually find equilibria in bimatrix games we refer to Eaves (1971), Kuhn (1961), Lemke and Howson (1964), Mangasarian (1964), Mangasarian and Stone (1964), Vorob'ev (1958) and Winkels(1979).

4.3. *Repeated games*

Repeated games are mostly considered under the average criterion. Looking for equilibria in a repeated game it is not difficult to see that an equilibrium in the one-step game immediately leads to a stationary equilibrium in the repeated game. Since the one-step game is simply a bimatrix game it has an equilibrium (ρ,σ). Considering ρ as stationary strategy for player I in the repeated game and similarly σ for player II, (ρ,σ) is an equilibrium consisting of stationary strategies in the repeated game. So we know that at least one equilibrium exists.

But there is more to say. For repeated games there is the following remarkable theorem, which is known as the Folk-theorem. It is known already for many years, but it's authorship is obscure. The Folk-theorem : the Nash equilibrium payoffs in the repeated game are the feasible individually rational payoff vectors in the one-step game. Feasible payoff vectors are payoff vectors that can actually occur for some pair of strategies (π_1,π_2). Observe that for repeated games a payoff vector (r_1,r_2) is feasible it it is a convex comtination of payoff vectors to pairs of pure actions.

It is individually rational if $r_k \geqq g_k$, for $k = 1,2$, where g_k is the value of the one-step k-zerosum game. The proof of this theorem is in principal quite simple. Having a feasible payoff vector (r_1,r_2) one can construct a sequence of pairs of pure actions (i_n,j_n), $n \in \mathbb{N}$, such that the corresponding average payoff vector converges to (r_1,r_2). Now $\pi_1 = (i_1,i_2,\ldots)$ and $\pi_2 = (j_1,j_2,\ldots)$ are pure Markov strategies for players I and II respectively, for which $g_k(\pi_1,\pi_2) = r_k$ for $k = 1,2$. Let ρ and σ be optimal mixed actions for player I and II respectively in the II-zerosum and the

I-zerosum game respectively. So $g_1(.,\sigma) \leqq g_1$ and $g_2(\rho,.) \leqq g_2$. Define π_1^* for player I by : at stage $n \in \mathbb{N}$ choose i_n if player II chose j_k for all $k < n$, else choose an action according to ρ. Define π_2^* analogously by combining π_2 and σ.

Then (π_1^*,π_2^*) is an equilibrium in the repeated game and for $k = 1,2$ $g_k(\pi_1^*,\pi_2^*) = (r_1,r_2)$. Suppose $\bar{\pi}_2$ is such that at some stage n player II does not choose j_n, whereas player I has chosen i_k for all $k < n$, then $g_2(\pi_1^*,\bar{\pi}_2) \leqq g_2 \leqq r_2$.

So each player threatens to keep his opponents income below g in case of doing something else than the equilibrium strategy. Hence (g_1,g_2) is known as the threat point.

For more about repeated games we refer to Aumann (1981) and Sorin (1986-b).

4.4. β-*Discounted stochastic games*

As we have seen zerosum β-discount stochastic games can be solved by finding a solution to the Shapley equation, in which matrix games play an important role. In non-zerosum β-discount stochastic games stationary equilibria are related with equilibria in bimatrix games in the following way (cf. Vrieze (1983)) :

A pair of stationary strategies (ρ,σ) with β-discounted rewards $v_1^\beta(\rho,\sigma)$ and $v_2^\beta(\rho,\sigma)$ to players I and II respectively is a β-discount equilibrium if and only if for each $s \in S$ (ρ_s,σ_s) is an equilibrium in the bimatrix game with payoff matrices $M_{1s}^\beta(v_1^\beta(\rho,\sigma))$ and $M_{2s}^\beta(v_2^\beta(\rho,\sigma))$ to players I and II respectively and with equilibrium payoff $(v_1^\beta(\rho,\sigma),v_2^\beta(\rho,\sigma))$. Here $M_{ks}^\beta(v_k^\beta(\rho,\sigma))$ is $M_s^\beta(v_k^\beta(\rho,\sigma))$ for the k-zerosum game (cf. 3.3).

This lemma tells us that finding a β-discount stationary equilibrium is equal to finding $\rho^* = (\rho_1^*,\rho_2^*,\ldots,\rho_z^*)$, $\sigma^* = (\sigma_1^*,\sigma_2^*,\ldots,\sigma_z^*)$ and, for $k = 1,2$, $x_k = (x_{k1},x_{k2},\ldots,x_{kz}) \in \mathbb{R}^z$ such that for all s, ρ_s and σ_s :

$$x_{1s} = \rho_s^* M_{1s}^\beta(x_1)\sigma_s^{*t} \geqq \rho M_{1s}^\beta(x_1)\sigma_s^{*t} \text{ and}$$

$$x_{2s} = \rho_s^* M_{2s}^\beta(x_2)\sigma_s^{*t} \geqq \rho_s^* M_{2s}^\beta(x_2)\sigma_s^{t}.$$

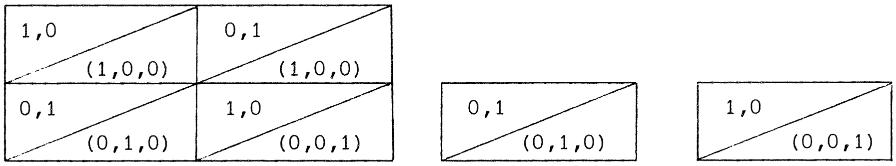
Note that for stationary strategies ρ and σ and $k = 1,2$ $v_k^\beta(\rho,\sigma)$ is the unique solution of the system of z equalities $x_s = \rho_s M_{ks}^\beta(x)\sigma_s$.

Fink (1964) was the first to show that stationary β-discount equilibria do exist. Proofs of existence of β-discount stationary equilibria can also be found in Takahashi (1964), Rogers (1969) and Sobel (1971). All these proofs are based on one or another fixed point theorem. Fink and Rogers use the fixed point theorem of Kakutani (1941); Takahashi uses a fixed point theorem of Fan (1952) and Glicksberg (1952); Sobel uses the Brouwer fixed point theorem, and his proof goes along the lines of the proof of Nash for equilibria in bimatrix games (cf. 4.2). Rogers' proof is based on the idea that a pair of stationary strategies (ρ^*,σ^*) is an equilibrium if and only if ρ^* is a best answer for player I against σ^* and σ^* is a best answer for player II against ρ^*. Define for stationary strategies σ of player II $B_1(\sigma) = \{\bar{\rho} : v_1^\beta(\bar{\rho},\sigma) = \max_\rho v_1^\beta(\rho,\sigma)\}$, the set of stationary best answers for player I against σ. Similarly define $B_2(\rho)$ for player II. Then a pair of strategies (ρ^*,σ^*) is a β-discount equilibrium if and only if $\rho^* \in B_1(\sigma^*)$ and $\sigma^* \in B_2(\rho^*)$. Now $B_1 \times B_2 : Y \times X \to 2^{Y\times X}$ where Y resp. X is the set of stationary strategies of player II resp. I. Rogers shows that the Kakutani fixed point theorem can be applied, giving the existence of a β-discount stationary equilibrium.

For an algorithm to actually compute β-discount stationary equilibria we refer to Breton et al. (1986), who extend work of Filar and Schultz (1986) on solving zerosum β-discount stochastic games with a special structure. The technique presented by Breton et al. can also be viewed as a natural consequence of a characterization of β-discount stationary equilibria by a system of linear inequalities, which is given in Sobel (1971). A characterization of β-discount stationary equilibria in terms of a solution of a related nonlinear complementarity problem is given in Vrieze (1983).

4.5. *Average stochastic games*

For average non-zerosum stochastic games it is well known that equilibria need not exist. Take for example (cf. 3.4) :



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However, it is still unknown whether or not average ε-equilibria always exist. For many special classes of stochastic games, i.e. stochastic games with special properties for the transitions or for the rewards, average (ε-)equilibria turned out to exist. In the light of the latest developments in this field it is likely that the question of existence of average ε-equilibria for general stochastic games can be shown in the near future.

In 4.6 we take a look at several special classes of stochastic games, for most of which it is known that they possess average ε-equilibria.

4.6. *Special stochastic games*

4.6.1. *Irreducible stochastic games*

A stochastic game is called irreducible if each pair of pure stationary strategies gives rise to an irreducible Markov chain, i.e. for all states s and t the play will move from s to t with probability 1.

Rogers (1969) was the first to examine games with this property and he was able to show that for these irreducible stochastic games average stationary equilibria do exist. He also showed that the irreducibility property could be weakened : if each pair of pure stationary strategies determines a Markov chain with a single ergodic subchain, then this implies the existence of an average stationary equilibrium. His proof makes use of a generalisation of the Kakutani (1941) fixed point theorem. Rogers (1969) also proves that average stationary equilibria can be characterized by being part of a solution to a related non-linear programming problem.

Stern (1975) was able to reduce the irreducibility condition some more.

He shows that average stationary equilibria exist in case there is a state s^* in the stochastic game such that a play starting in any state s will pass through s^* with positive probability.

Further, Federgruen (1978) shows that average stationary equilibria exist under any of the following properties :

i) there is some set of states A and some number N such that for all pairs of stationary strategies (ρ,σ) and any starting state s the expected number of stages to get from s to A is at most N.

ii) there is some number N such that, given states s and t and a stationary strategy ρ for player I there exists a stationary strategy σ for player II such that the expected number of stages to get from s to t under ρ and σ is at most N.

Federgruen (1978) also shows that the average stationary equilibria can be found as limits of β-discount stationary equilibria for β tending to 1. In fact for each irreducible stochastic game any converging sequence of β-discount stationary equilibria, for β going to 1,leads to an average stationary equilibrium.

4.6.2. *Single-controller stochastic games*

These are stochastic games in which one of the players controls the transitions, i.e. $p(s,i,j) = p(s,j)$ for all s,i,j if player II is the controller. Parthasarathy and Raghavan (1981) show that for these games average stationary strategies can be found as limits of converging sequences of β-discount stationary equilibria. Parthasarathy and Raghavan also show that single-controller stochastic games possess the ordered field property (cf. 2.7) in the β-discount as well as the average case.

Filar (1984) shows that for single-controller stochastic games the set of average stationary equilibria can be constructed from a finite number of extreme equilibrium strategies for player I and a finite number of pseudo-extreme equilibrium strategies for player II, the controller of the game. The (pseudo-)extreme equilibrium strategies can be constructed by finite algorithms.

Average equilibria, as well as β-discount equilibria, can be obtained, for these games, from an optimal solution of an appropriately constructed quadratic program (cf. Filar (1986)).

4.6.3. *SER-SIT stochastic games*

SER-SIT stochastic games are stochastic games with separable rewards, i.e. $r_k(s,i,j) = c_k(s) + r_k(i,j)$ for all s,i,j and $k = 1,2$, and state independent transitions, i.e. $p(t \mid s,i,j) = p(t \mid s',i,j)$ for all $s,s',t \in S$ and all i,j. Here it is assumed without loss of generality that all matrices M_s, $s \in S$, have the same size. These games were introduced by Parthasarathy et al. (1984). They show that β-discount stationary equilibria, as well as average stationary equilibria, exist which correspond with equilibria in related bimatrix games. Those bimatrix games are $[r_1(i,j) + \beta \sum_{t=1}^{z} p(t \mid i,j)c_1(t),\ r_2(i,j) + \beta \sum_{t=1}^{z} p(t \mid i,j)c_2(t)]$ for the β-discount case, and the same with $\beta = 1$ for the average case.
For SER-SIT stochastic games a limit of β-discount stationary equilibria leads to an average stationary equilibrium.

Furthermore SER-SIT stochastic games possess the ordered field property and Parthasarathy et al. (1984) give examples that neither the SER property alone nor the SIT property alone guarantee that the ordered field property holds.

4.6.4. *ARAT stochastic games*

Raghavan et al. (1985) examined ARAT stochastic games : stochastic games with additive rewards, i.e. $r_k(s,i,j) = r_k(s,i) + r_k(s,j)$, and additive transitions, i.e. $p(t \mid s,i,j) = p(t \mid s,i) + p(t \mid s,j)$, for all s,i,j and $k = 1,2$. For these games the existence of average ε-equilibria is unknown but of course β-discount stationary equilibria do exist. Raghavan et al. (1985) show that always β-discount stationary equilibria exist for which each player in each state has to mix only two pure actions. An example is also given that this cannot be sharpened any further, pure β-discount stationary equilibria do not need to exist.

ARAT stochastic games possess the ordered field property in the β-discount case, and also in the average zerosum case.

4.6.5. *Stochastic games with one player controlling the rewards*

These are stochastic games with $r_k(s,i,j) = r_k(s,i)$ for all s,i,j and k = 1,2. Vrieze et al. (1985) show that for β-discount games of this type with just two states the ordered field property holds. We sketch the proof. If player I fixes a stationary strategy $\rho = (\rho_1,\rho_2)$ then player II faces a β-discount Markov decision problem, β-MDP(ρ), for which it is well known that player II has pure optimal stationary strategies. Let $C(\rho_1)$ denote the carrier of ρ_1, i.e. the set of actions on which ρ_1 puts positive weight, and $C(\rho_2)$ the carrier of ρ_2. For a stationary strategy $\sigma = (\sigma_1,\sigma_2)$ of player II β-MDP(σ), $C(\sigma_1)$, $C(\sigma_2)$ have analogous meaning. Further let $F_1(C(\rho_1) \times C(\rho_2)) = \{\sigma$: all elements of $C(\rho_1) \times C(\rho_2)$ are optimal stationary strategies for player I in β-MDP(σ)$\}$, and let $F_2(C(\sigma_1) \times C(\sigma_2))$ be defined analogously. Then a pair of stationary strategies (ρ,σ) is a β-discount equilibrium if and only if $\rho \in F_2(C(\sigma_1) \times C(\sigma_2))$ and $\sigma \in F_1(C(\rho_1) \times C(\rho_2))$. Vrieze et al. (1985) prove that in case all data of the game are rational, then $F_1(C(\rho_1) \times C(\rho_2))$ is either empty or a polytope with rational extreme points and $F_2(C(\sigma_1) \times C(\sigma_2))$ is either empty or the union of at most three polytopes with rational extreme points. The existence of β-discount stationary equilibria (cf. Fink (1964)) gives that there exist $(\bar{\rho},\bar{\sigma})$ with $\bar{\rho} \in F_2(C(\bar{\sigma}_1) \times C(\bar{\sigma}_2))$ and $\bar{\sigma} \in F_1(C(\bar{\rho}_1) \times C(\bar{\rho}_2))$. From this and the above remark on polytopes with rational extreme points, Vrieze et al. (1985) derive that there exists a rational β-discount equilibrium (ρ^*,σ^*).

4.6.6. *Repeated games with absorbing states*

Zerosum stochastic games of this type have first been examined very thoroughly by Kohlberg (1974), who was able to applicably describe average ε-optimal behaviour strategies for any game of this type. His work was a big step forward in solving the problem of existence of average value and average ε-optimal strategies.

Sorin (1986a) examined a non-zerosum repeated game with absorbing states. Whereas for zerosum stochastic games $g = \lim_{\beta\uparrow 1}(1-\beta)\, v^\beta$, the example of Sorin (1986a) shows that such a relation need not hold for the average Nash equilibrium payoffs and the β-discount Nash equilibrium payoffs (cf. 2.4.1); so the relation $g\text{-Nep} = \lim_{\beta\uparrow 1}(1-\beta)(\beta - \text{Nep})$ need not hold. Hence, Sorin concludes that the average game cannot be properly approximated by a β-discount game, for any $\beta \in (0,1)$.

The work of Sorin (1986a) and Kohlberg (1974) inspired Vrieze and Thuijsman (1986) to investigate the class of non-zerosum repeated games with absorbing states. They succeeded in proving the existence of average ε-equilibria for these games. Like in the zerosum case behaviour strategies are indispensable and like in repeated games threats are of importance to establish the average ε-equilibria. Vrieze and Thuijsman (1986) show how an average ε-equilibrium can be constructed from converging sequences of β-discount stationary equilibria. Not always by taking the limit; the way the transition probabilities change as β goes to 1, sometimes has to be used to construct a second strategy for one of the players. In some cases one of the players has to play the two limit strategies in a behavioural way to form an ε-equilibrium with a certain strategy of the other player, and by making use of threats, which can also be of the behaviour type.

5. THE BANKROBBERY GAME

We give a solution for the game introduced in 2.2 :

State 1:

2,0 / (1,0,0)	1,10 / (9/10,0,1/10)
1,0 / (1,0,0)	1,10 / (9/10,1/10,0)

State 2:

2,-1 / (0,1,0)

State 3:

0,10 / (0,0,1)

5.1. *The I-zerosum stochastic game*

We examine the zerosum stochastic game which rises if we only consider the payoffs to player I, starting of with the β-discount game.

Denote the β-discount value by $(v_1^\beta, v_2^\beta, v_3^\beta)$. Then it is clear that $v_2^\beta = 2/(1-\beta)$ and $v_3^\beta = 0$. v_1^β can be computed from the Shapley equation (cf. 3.3) :

$$v_1^\beta = \text{val}\begin{bmatrix} 2+\beta v_1^\beta & 1+(9/10).\beta v_1^\beta \\ 1+\beta v_1^\beta & 1+(9/10).\beta v_1^\beta + (1/10).2\beta/(1-\beta) \end{bmatrix}$$

From this one can derive : $v_1^\beta = (10-6\beta)/((1-\beta)(10-7\beta))$. The unique β-discount stationary optimal strategies are $\rho^\beta = (\beta/(10-7\beta),(10-8\beta)/(10-7\beta))$ for player I and $\sigma^\beta = (\beta/(5-4\beta),(5-5\beta)/(5-4\beta))$ for player II.

Knowing that the average value $g = \lim_{\beta\uparrow 1}(1-\beta)v^\beta$ (cf. 3.4) we get for this game $g = (4/3,2,0)$. We also note that $\rho^1 = \lim_{\beta\uparrow 1} \rho^\beta = (1/3,2/3)$ is an average optimal stationary strategy for player I in the I-zerosum stochastic game. However $\sigma^1 = \lim_{\beta\uparrow 1} \sigma^\beta = (1,0)$ is no average optimal stationary strategy for player II in the I-zerosum stochastic game. In fact player II has not even got an average ε-optimal Markov strategy in the I-zerosum game (for ε small enough).

5.2. *The II-zerosum stochastic game*

This time we only look at the payoffs to player II, again starting with the β-discount stochastic game.

The β-discount value of the II-zerosum game is denoted $(\bar{v}_1^\beta,\bar{v}_2^\beta,\bar{v}_3^\beta)$. We immediately have $\bar{v}_2^\beta = -1/(1-\beta)$ and $\bar{v}_3^\beta = 10/(1-\beta)$. Again $\bar{v}_1^\beta$ can be obtained from the Shapley equation :

$$\bar{v}_1^\beta = \text{val}\begin{bmatrix} \beta\bar{v}_1^\beta & 10+(9/10).\ \beta\bar{v}_1^\beta + (1/10).10\beta/(1-\beta) \\ \beta\bar{v}_1^\beta & 10+(9/10).\ \beta\bar{v}_1^\beta + (1/10).(-\beta)/(1-\beta) \end{bmatrix}$$

This gives us : $\bar{v}_1^\beta = 0$ for $\beta \in (100/101,1)$.

For $\beta \in (100/101,1)$ the unique β-discount optimal stationary strategy for player II is $\bar{\sigma}^\beta = (1,0)$.

For player I and $\beta \in (100/101,1)$ any stationary strategy $\bar{\rho}^\beta$ in co$\{((101\beta-100)/11\beta,(100-90\beta)/11\beta),(0,1)\}$ is β-discount optimal. Here "co" denotes "convex hull of".

For $\beta = 100/101$: $\bar{v}_1^\beta = 0$; for player II any stationary strategy in co$\{(1,0),(0,1)\}$ is β-discount optimal, for player I $(0,1)$ is the unique β-discount optimal stationary strategy.

For $\beta < 100/101$: $\bar{v}_1^\beta = (100-101\beta)/((1-\beta)(10-9\beta))$; for player II, as well as for player I, (0,1) is the unique β-discount optimal stationary strategy.

The average value of the II-zerosum stochastic game is $\bar{g} = \lim_{\beta\uparrow 1}(1-\beta)\bar{v}^{\beta} = (0,-1,10)$. An average optimal stationary strategy for player II is $\bar{\sigma}^1 = (1,0)$; any stationary strategy $\bar{\rho}^1 \in \mathrm{co}\{(1/11,10/11),(0,1)\}$ is average optimal for player I.

5.3. *The non-zerosum stochastic game*

Let us now look for equilibria. First of all in the β-discount stochastic game.

For $\beta \leqq 100/101$ one can verify that $((0,1),(0,1))$ is a β-discount stationary equilibrium corresponding with the β-Nep $((10-8\beta)/((1-\beta)(10-9\beta)), (100-101\beta)/((1-\beta)(10-9\beta)))$.

For $\beta > 100/101$ the unique β-discount stationary equilibrium is $(\bar{\rho}^{\beta},\sigma^{\beta})$ with $\bar{\rho}^{\beta} = ((101\beta-100\beta)/11\beta,(100-90\beta)/11\beta)$ and $\sigma^{\beta} = (\beta/(5-4\beta),(5-5\beta)/(5-4\beta))$; the corresponding β-Nep is $(v_1^{\beta}(\bar{\rho}^{\beta},\sigma^{\beta}),v_2^{\beta}(\bar{\rho}^{\beta},\sigma^{\beta})) = ((10-6\beta)/((1-\beta)(10-7\beta)),0) = (v_1^{\beta},v_2^{\beta})$.

Now, consider the limits $\bar{\rho}^1 = \lim_{\beta\uparrow 1} \bar{\rho}^{\beta} = (1/11,10/11)$, $\sigma^1 = \lim_{\beta\uparrow 1} \sigma^{\beta} = (1,0)$,

$\lim_{\beta\uparrow 1}(1-\beta)(v_1^{\beta}(\bar{\rho}^{\beta},\sigma^{\beta}),v_2^{\beta}(\bar{\rho}^{\beta},\sigma^{\beta})) = (4/3,0)$.

Note that $(4/3,0) = (g_1,g_2)$.

$g_1(\bar{\rho}^1,\sigma^1) = 12/11 < 4/3 = g_1$ and $g_2(\bar{\rho}^1,\sigma^1) = 0 = g_2$.

Hence $(\bar{\rho}^1,\sigma^1)$ is no average stationary equilibrium; player I can do better against σ^1.

Observe that for $(\bar{\rho}^1,\sigma^1)$ state 1 is recurrent whereas for $(\bar{\rho}^1,\sigma^{\beta})$ state 1 is transient for all $\beta < 1$.

Also note that for $\sigma^2 = (0,1)$ $g_1(\bar{\rho}^1,\sigma^2) = 20/11 > 4/3$ and $g_2(\bar{\rho}^1,\sigma^2) = 0$.

Let σ^* be an average ε-optimal behaviour strategy for player II in the following modified I-zerosum stochastic game. Here $8/11 = g_1(\bar{\rho}^1,\sigma^2) - g_1(\bar{\rho}^1,\sigma^1)$.

2+8/11 (1,0,0)	1 (9/10,0,1/10)
1+8/11 (1,0,0)	1 (9/10,1/10,0)

1

2 (0,1,0)

2

0 (0,0,1)

3

σ^* can be constructed such that when playing against $\bar{\rho}^1$ any play started in state 1 will move to state 2 or state 3 with probability 1.
Now one can verify that $(\bar{\rho}^1,\sigma^*)$ is an average ε-equilibrium with g-Nep $(g_1(\bar{\rho}^1,\sigma^*),g_2(\bar{\rho}^1,\sigma^*)) = (20/11,0)$.

For a general approach of how to construct average ε-equilibria in non-zerosum repeated games with absorbing states, we refer to Vrieze and Thuijsman (1986).

6. CONCLUDING REMARKS

With respect to non-zerosum stochastic games, as we defined them in the beginning of this chapter, there are still a lot of topics which ask for further research. To solve the problem of existence of average ε-equilibria is just one of them. A lot of work could be done in finding suitable algorithms to compute (ε-)equilibria in the β-discount as well as the average case. To compute the rewards if behaviour strategies are being used is also mostly a rather difficult task. There are much more of such questions.

The survey given in this chapter is far from complete. Very little has been said about algorithmic aspects, or about characterizations of special types of equilibria (cf. Couwenbergh (1977), Groenewegen and Wessels (1979), Wessels (1981)). Also a lot of work has been done on non-zerosum stochastic games other than with finite state and action spaces. Concerning such games we mention in alphabetic order Nowak (1985), Parthasarathy (1982), Parthasarathy and Raghavan (1975), Parthasarathy and Sinha (1986), Sobel (1973) and Tijs (1980).

Finally we would like to refer to the survey papers of Parthasarathy and Stern (1977) and Raghavan (1984) which, beside dealing with a lot of aspects of stochastic games, have very comprehensive bibliographic listings.

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CHAPTER VI

FROM DECISION MAKING UNDER UNCERTAINTY TO GAME THEORY

by Peter Wakker

1. INTRODUCTION

From a mathematical point of view many results from game theory and decision making under uncertainty are equivalent. An example is the characterization, as the class of "balanced" games, of the class of cooperative games with side-payments which have nonempty core. This was found by Shapely (1967); earlier Bondareva (1963, in Russian) had obtained this result; see also Driessen (1985, section 2.8). In Huber (1981, Lemma 10.2.2) the same result, obtained independently, is given for the context of decision making under uncertainty. Many other results have been formulated for one of the two contexts, but seem to be as interesting when formulated for the other context. One such example, not elaborated in this paper, is the theory of "belief functions" of Shafer (1976), formulated for the context of decision making under uncertainty. We think that notions such as the "degree of internal conflict" of a belief function, as developed by Shafer, are of utmost interest when studied for game theory. For a concise introduction into the basic concepts of Shafer's theory, see Zang (1986, section 1).

This paper presents new approaches to several topics in game theory. The obtained results have in common that they have been derived, by simple translation algorithms, from results on probability theory and decision making under uncertainty. Section 5 will show how this was done. Further in section 5 proofs will be indicated.

The aim of this paper is to show the usefulness of the adopted translation algorithms.

2. ORDERING COALITIONS IN COOPERATIVE GAME THEORY

First we present the basic definitions of the theory of cooperative games with side payments. Let $N = \{1,\dots,n\}$ be a nonempty finite set of *players*, and 2^N the set of *coalitions*. A function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0, S \supset T \Rightarrow v(S) \geqq v(T)$, and $v(N) = 1$ is called *characteristic function*; the second (monotonicity) condition, and the third (normalizing) condition, are not generally assumed in literature, but for convenience will be assumed throughout this paper. The quantity $v(S)$ may designate for instance the power, or earnings, or (negative) costs of a coalition S, or the number of publications of S in the International Journal of Game Theory; in this paper $v(S)$ will be called the *worth* of the coalition S. An element $x = (x_1,\dots,x_n) \in \mathbb{R}_+^n$ is an *allocation*, and is interpreted as a function, assigning x_j to player j, for all j. In this paper $\mathbb{R}_+$ is the set containing 0 and all positive real numbers. The quantity x_j may for instance stand for money. If $\Sigma_{j=1}^n x_j = v(N)$, x may be interpreted as a division of the worth of the "grand" coalition $\{1,\dots,n\}$, and x is called *efficient*. A central question in the theory of cooperative games with side payments is the question which efficient allocation is "fair" for a characteristic function v. The usual procedure to determine this is to compare the amount $x(S) := \Sigma_{j\in S} x_j$, allocated to the coalition S, with the worth $v(S)$ of the coalition S, and,for instance, to take as a criterion that every $x(S)$ should be at least as large as the worth of the coalition. In that case x is called a *core* allocation.

This paper will propose criterions of a different character. The idea of our criterions will be that the central notions to be considered are the *orderings* of coalitions as induced by the worths and allocations, and not the worths and allocations themselves. As an example where this may be natural think of the many cases, e.g. in politics, where an (inefficient) allocation (5,5,...,5) over a set of persons with equal worth is preferred over an allocation (6,7,6,7,...), simply because the second allocation would induce "unjust" inequalities, and tensions. As a second example think of definitions of wealth which say that a person is rich if she (or he) belongs to the 20 percent of most wealthy persons in her country. Again it is the ordering induced by allocated money which is relevant, not the absolute amounts of money.

Let us now consider some criteria of the new kind. They all express the idea that more worthy coalitions should get allocated more.

DEFINITIONS 2.1. An allocation x is :
Almost agreeing with v if, for all coalitions S,T,
$[v(S) \geqq v(T) \Rightarrow x(S) \geqq x(T)]$;
Strictly agreeing with v if, for all coalitions S,T,
$[v(S) > v(T) \Rightarrow x(S) > x(T)]$;
Agreeing with v if, for all coalitions S,T,
$[v(S) \geqq v(T) \Leftrightarrow x(S) \geqq x(T)]$.

The first criterion above might be called "socialistic" since it allows for the occurrence of coalitions S,T with $x(S) = x(T)$ while $v(S) > v(T)$, whereas $v(S) = v(T)$ will always imply $x(S) = x(T)$; thus equality is increased by it. The second criterion might be called "capitalistic" since it allows for the occurrence of coalitions S,T with $x(S) > x(T)$ while $v(S) = v(T)$, whereas $v(S) > v(T)$ will always imply $x(S) > x(T)$. Obviously an allocation is agreeing if and only if it is both strictly agreeing and almost agreeing. Also one elementarily verifies that x is almost agreeing with v if and only if, for all coalitions S,T, $[x(S) > x(T) \Rightarrow v(S) > v(T)]$, and strictly agreeing with v if and only if for all coalitions S,T, $[x(S) \geqq x(T) \Rightarrow v(S) \geqq v(T)]$.

It will be observed that not for every characteristic function v there exist agreeing allocations x. For example let $N = \{1,2,3\}$, and let v assign 1/3 to every one-player coalition, 1/2 to $\{1,2\}$, and 2/3 to every other two-player coalition. Then, to be agreeing, x will have to assign the same to every one-player coalition, which will imply $x\{1,2\} = x\{2,3\}$; however $v\{1,2\} < v\{2,3\}$ should imply $x\{1,2\} < x\{2,3\}$. The characteristic function just described does not satisfy the following condition (set $S = \{1\}$, $T = \{3\}$, $V = \{2\}$ in the definition below):

DEFINITION 2.2. The characteristic function v is *ordinally additive* if, for all coalitions S,T,V with $S \cap V = \emptyset = T \cap V$:
$v(S) \geqq v(T) \Leftrightarrow v(S \cup V) \geqq v(T \cup V)$.

It is straightforwardly verified that this condition is necessary for the existence of an agreeing allocation. Still, it turns out not to be sufficient, as the following example shows.

EXAMPLE 2.3. (Kraft, Pratt & Seidenberg). Let $N = \{1,2,3,4,5\}$, and let $v(\emptyset) = 0$, $v\{1\} = 2/32$, $v\{2\} = 3/32$, $v\{3\} = 4/32$, $v\{1,2\} = 5/32$, $v\{1,3\} = 6/32$, $v\{4\} = 7/32$, $v\{1,4\} = 8/32$, $v\{2,3\} = 9/32$, $v\{5\} = 10/32$, $v\{1,2,3\} = 11/32$, $v\{2,4\} = 12/32$, $v\{3,4\} = 13/32$, $v\{1,5\} = 14/32$, $v\{1,2,4\} = 15/32$, $v\{2,5\} = 16/32$, $v\{1,3,4\} = 17/32$, $v\{3,5\} = 18/32$, $v\{2,3,4\} = 19/32$, $v\{1,2,5\} = 20/32$, $v\{1,3,5\} = 21/32$, $v\{4,5\} = 22/32$, $v\{1,2,3,4\} = 23/32$, $v\{1,4,5\} = 24/32$, $v\{2,3,5\} = 25/32$, $v\{1,2,3,5\} = 26/32$, $v\{2,4,5\} = 27/32$, $v\{3,4,5\} = 28/32$, $v\{1,2,4,5\} = 29/32$, $v\{1,3,4,5\} = 30/32$, $v\{2,3,4,5\} = 31/32$, $v(N) = 1$.

It is straightforwardly checked that this v is a characteristic function which satisfies ordinal additivity. Still, no agreeing allocation x exists since the inequalities $x\{1\} + x\{3\} < x\{4\}$, $x\{1\} + x\{4\} < x\{2\} + x\{3\}$, $x\{3\} + x\{4\} < x\{1\} + x\{5\}$, $x\{2\} + x\{5\} < x\{1\} + x\{3\} + x\{4\}$, when added up, reveal a contradiction.

In the above example there does exist an almost agreeing efficient allocation, viz. (1/16,2/16,3/16,4/16,6/16). There do exist characteristic functions for which no almost agreeing efficient allocation exists, and characteristic functions for which no strictly agreeing allocation exists, whereas these characteristic functions do satisfy ordinal additivity. Further it can be seen that for all cooperative games with side payments with less than five players, ordinal additivity is sufficient for the existence of agreeing allocations. For all cooperative games with side payments with less than six players ordinal additivity is sufficient for the existence of an almost agreeing efficient allocation. The reader may want to check these facts by writing a computer program on his personal computer which checks all cases.

The necessary and sufficient conditions for the existence of the several kinds of agreeing allocations can be obtained by standard applications of theorems of the alternative, (see for instance Scott, 1964), and are as follows, with $x \leqq\leqq y$ if $x_j \leqq y_j$ for all j, $x >> y$ if $x_j > y_j$ for all j, and $x \underset{\neq}{>} y$ if $x_j \geqq y_j$ for all j, and $x \neq y$.

THEOREMS 2.4. *There exists an almost agreeing efficient allocation if and only if : For every pair of sequences of coalitions* $(S_1,\ldots,S_n)$ *and* $(T_1,\ldots,T_n)$ *for which every player occurs in more coalitions in the left sequence than in the right*

$$not(v(S_1),\ldots,v(S_n)) \leqq\leqq (v(T_1),\ldots,v(T_n)). \tag{2.1}$$

There exists a strictly agreeing efficient allocation if and only if : For every pair of sequences of coalitions $(S_1,\ldots,S_n)$ *and* $(T_1,\ldots,T_n)$ *for which every player occurs in at least as many coalitions in the left sequence as in the right*

$$not(v(S_1),\ldots,v(S_n)) >> (v(T_1),\ldots,v(T_n)). \tag{2.2}$$

There exists an agreeing efficient allocation if and only if : For every pair of sequences of coalitions $(S_1,\ldots,S_n)$ *and* $(T_1,\ldots,T_n)$ *for which every player occurs in the same number of coalitions in the left sequence as in the right*

$$not(v(S_1),\ldots,v(S_n)) \underset{\neq}{>} (v(T_1),\ldots,v(T_n)). \tag{2.3}$$

□

Obviously the third condition in the theorem has to imply ordinal additivity of v. Note that the only property of v, used in our analysis, has been the way v orders the coalitions. Thus we might also have taken an ordering of the coalitions, instead of v, as primitive in our analysis. Note that without the efficiency restriction there always exists an almost agreeing allocation : (0,...,0). For agreeing allocations, and strictly agreeing allocations, $v\{1,\ldots,n\}$ is positive, so x can always be normalized, and the requirement of efficiency in the above theorem does not induce any restriction, so might have been omitted.

We end this section with a conjecture : there exists a characteristic function which is ordinally additive, which has both an almost agreeing and a strictly agreeing allocation but no agreeing one.

3. BANKRUPTCY PROBLEMS

Let $n \in \mathbb{N}$ be fixed, $n \geqq 3$. Let $E \in \mathbb{N}$ be fixed, and let $d = (d_1,\ldots,d_n) \in \mathbb{N}_0^n$. E is an amount of money, to be divided among n players (or claimants) 1,...,n where each player j has advanced a claim of d_j. For any d, by d_+ we denote the total amount $\Sigma\, d_j$ of claims. A *division rule* $f : \mathbb{N}_0^n \to \mathbb{R}^n$ is a function which assigns to every claim $d = (d_1,\ldots,d_n)$ a sequence of proportions $(f_1(d),\ldots,f_n(d))$, with $f_j(d) \geqq 0$ for all j, and $\Sigma\, f_j(d) = 1$, such that player j will receive a portion $f_j(d) \times E$ of the amount E. Obviously one might think of other interpretations, e.g. where d_j reflects the salary of a person j, and $f_j(d)$ the tax which the person is to pay; also d_j may stand for investment, one-player-coalition-worth, etc. Our set-up differs from the usual set-ups such as Aumann, R.J. & M.Maschler (1985), Moulin (1985a,b), Curiel, I., M. Maschler & S.H. Tijs (1986), and Young (1987) in considering only natural numbers as claims (and amounts) to be divided, and in leaving out of the analysis variability of the amount E to be divided.

We shall now proceed to consider some conditions for division rules.

DEFINITION 3.1. We call f *monotone* if, for all i and d :

$$f_i(d_1,\ldots,d_i+1,\ldots,d_n) > f_i(d_1,\ldots,d_i,\ldots,d_n).$$

So if a player can increase her (or his) claim, it will give her a larger portion of the amount E.

DEFINITION 3.2. We call a player i *uninvolved* for the division rule f if, for all $j \neq i \neq k$, and all d :

$$f_i(d_1,\ldots,d_j+1,\ldots,d_k,\ldots,d_n) = f_i(d_1,\ldots,d_j,\ldots,d_k+1,\ldots,d_n).$$

So if a player i is uninvolved, she has no interest in a replacement of part of the claim of player j to another player k. Her proportion $f_i(d)$ will depend only on her own claim d_i and the total claim $(d_+ - d_i)$ of the other players, as is easily verified. It protects player i against a manipulation of the remaining players to increase the sum of their shares by redistributing amongst each other the sum of their claims. Moulin (1985a) introduced the condition that all players shall be uninvolved in a related context and called it "No Advantageous Reallocation".

DEFINITION 3.3. We call a pair of players i,j *proportionally uninvolved* if, for all $i \neq k \neq j$:

$$\frac{f_i(d_1,\ldots,d_k+1,\ldots,d_n)}{f_j(d_1,\ldots,d_k+1,\ldots,d_n)} = \frac{f_i(d_1,\ldots,d_k,\ldots,d_n)}{f_j(d_1,\ldots,d_k,\ldots,d_n)}$$

where one denominator being zero is to imply that the other denominator is zero too; in the presence of monotonicity that can only happen if $d_j = 0$.

So then the proportion of the portions that player i and j receive from E depends only on d_i and d_j, and is independent of the other claims. This condition is somewhat stronger (in also restricting f_i/f_j if $f_i + f_j$ varies) than the consistency property as introduced in Kolm (1976, in a context with varying number of players and nonrational claims and amounts E); see also the consistency condition in Moulin (1985b). Now we characterize the division rules with the above properties.

THEOREM 3.4. *For a division rule* f *the following two statements are equivalent :*

(i) There exist nonnegative constants $\gamma_1,\ldots,\gamma_n$, summing to one, and a nonnegative constant λ, such that for all i :

$$f_i : d \mapsto \frac{(\lambda\gamma_i + d_i)}{(\lambda + d_+)} .$$

(ii) The division rule f satisfies monotonicity, every player is uninvolved, and every pair of players is proportionally uninvolved. □

Note that f_i as in (i) above can be considered to be a convex combination of the amount γ_i that player i would receive if no player would have claimed anything, and d_i/d_+, the share of the total amount of claims that has been advanced by player i, with weights respectively λ and d_+. An indication of a full proof is provided in subsection 5.2. Let us just sketch here a way of proof. It is straightforward that statement (i) above implies (ii). So we assume (ii), and derive (i). First one determines the constants $\gamma_1,\ldots,\gamma_n$ as $f_1(0,\ldots,0),\ldots,f_n(0,\ldots,0)$. Next one calculates λ from $f_1(1,0,\ldots,0) = (\lambda\gamma_1 + 1)/(\lambda + 1)$. Note that the division rule as defined in (i) above is one division rule with the mentioned values $f_1(0,\ldots,0),\ldots,f_n(0,\ldots,0)$ and $f_1(1,0,\ldots,0)$, satisfying all conditions of (ii). Finally the most involved part of the proof is to demonstrate that the above-mentioned values of f, together with the mentioned conditions, uniquely determine all values f(d) with $d_+ = 1$, next those with $d_+ = 2$; by induction with respect to d_+, the uniqueness of f(d) follows for all d. In this the monotonicity condition serves to prevent that certain equalities will reduce to the trivial 0 = 0.

4. BETTER AND WORSE ALLOCATIONS

As in the previous sections, we consider in this section the question of how to choose between several possible allocations $(x_1,\ldots,x_n)$ over n players. And, as in section 2, a characteristic function v will occur in our analysis. Still the approach of this section will be different, and in Theorem 4.2 the status of observability of v will differ from the usual set-up in the theory of cooperative games with side payments.

Let us first sketch the approach by means of "Choquet integrals" , central for this section. For simplicity of exposition we shall assume that an arbitrator will finally decide which of a set of available allocations to choose.

4.1. *The choquet-integral-approach*

The approach of this subsection will be split up in six stages.

Stage 1. The arbitrator concentrates for a moment on one available allocation x.

Stage 2. For this allocation x, the arbitrator takes a permutation π on $1,\ldots,n$ such that : $x_{\pi(1)} \geqq x_{\pi(2)} \geqq \ldots \geqq x_{\pi(n)}$. So $\pi(1)$ is the richest player under allocation x, $\pi(2)$ the one-after-the-richest-player, etc. Note that we have not specified the way in which equally-rich players are to be ordered according to π. They may be ordered in any arbitrary way, the approach sketched in the sequel will be such that this ordering is immaterial.

Stage 3. The players will enter, one by one, a room where the arbitrator is. First the richest player $\pi(1)$ enters, then $\pi(2)$, etc.

Step 3.1. After entrance of player $\pi(1)$ the arbitrator pays to $\pi(1)$ the amount $x_{\pi(1)} - x_{\pi(2)}$ that $\pi(1)$ receives more than player $\pi(2)$.

Step 3.2. Next player $\pi(2)$ enters the room, and the arbitrator pays to $\{\pi(1),\pi(2)\}$ the amount $x_{\pi(2)} - x_{\pi(3)}$ that $\pi(1)$ and $\pi(2)$ still are to receive more than player $\pi(3)$.

...

Step 3.i. Next player $\pi(i)$ enters the room, and the arbitrator pays to the present players $\pi(1),\ldots,\pi(i)$ the amount $x_{\pi(i)} - x_{\pi(i+1)}$ that the present players still are to receive more than player $\pi(i+1)$.

...

Step 3.n. Finally player $\pi(n)$ enters, all players are present now, and get payed the remaining amount $x_{\pi(n)}$.

Stage 4. Now that the payment in stage 3 has been fixed for every step, at every step the payment is valued by its product with the worth of the involved group of players.

Stage 5. The allocation x is valued by adding up all valuations of Stage 4, to give, with $x_{\pi(n+1)} := 0$:

$$\Sigma_{i=1}^{n}[(x_{\pi(i)} - x_{\pi(i+1)}) \times v\{\pi(1),\ldots,\pi(i)\}] \tag{4.1}$$

If we consider an allocation x as a function, assigning x_j to every player j, then the value in (4.1) is the *Choquet integral* of x with respect to the characteristic function v, see for instance Wakker (1986a, formula VI.2.5). Indeed, if v happens to be "additive", then (4.1) reduces to the usual integral.

Stage 6. For all available allocations a valuation is determined as it was for x above. Then the allocation with maximal valuation is chosen. If there are more allocations where the maximal valuation is attained, from these an arbitrary choice is made. If the supremum value of the valuations is not attained by any allocation, then some allocation is chosen which is close enough to the supremum in some sense. If the set of available allocations is compact, then the maximum will always be attained for some allocation.

4.2. *A characterization by ordering allocations*

In this section we characterize the approach of subsection 4.1. The method of characterization will differ from that of section 2. In this section we assume that the arbitrator takes a binary ("preference") relation $\succcurlyeq$ on $\mathbb{R}^n_+$, the set of all allocations. Here $x \succcurlyeq y$ means that the arbitrator would be willing to choose x if only x and y were available, i.e. she (or he) considers x at least as good as y. Next we consider conditions which will characterize the approach of subsection 4.1.

The binary relation $\succcurlyeq$ is a *weak order* if it is *complete* (i.e. for all x,y in $\mathbb{R}^n_+$: $x \succcurlyeq y$ or $y \succcurlyeq x$) and *transitive* (i.e. for all x,y,z in $\mathbb{R}^n_+$: if $x \succcurlyeq y$ and $y \succcurlyeq z$ then $x \succcurlyeq z$). As usual we write $x \succ y$ if $x \succcurlyeq y$ and not $y \succcurlyeq x$, $x \approx y$ if $x \succcurlyeq y$ and $y \succcurlyeq x$, $x \preccurlyeq y$ if $y \succcurlyeq x$, and $x \prec y$ if $y \succ x$. Further $\succcurlyeq$ is *strictly monotonic* if, for all allocations x,y,$[x > y \Rightarrow x \succ y]$, and $\succcurlyeq$ is *continuous* if, for all allocations y, the sets $\{x \in \mathbb{R}^n_+ : x \succcurlyeq y\}$ and $\{x \in \mathbb{R}^n_+ : x \preccurlyeq y\}$ are closed.

We call a pair of allocations x,y *comonotonic* if for no players i,j simultaneously $x_i > x_j$ and $y_j > y_i$. This is exactly the case where in Stage 2 of subsection 4.1 there can be chosen a same permutation π for x and y, see Wakker (1986a, Lemma VI.3.5, (i) $\Leftrightarrow$ (iii)). A set of allocations is *comonotonic* if every pair of allocations in the set is comonotonic. The main characterizing condition will be :

DEFINITION 4.1. The binary relation $\succcurlyeq$ satisfies *comonotonic independence* if for all comonotonic x,y,z and $\alpha \in (0,1)$ we have :

$$[x \succ y \Rightarrow \alpha x + (1-\alpha) z \succ \alpha y + (1-\alpha) z].$$

With this we get :

THEOREM 4.2. *For the binary relation ≧ on the set of allocations the following two statements are equivalent :*

(i) There exists a characteristic function v such that, for all allocations x,y, x ≧ y if and only if the Choquet integral of x is at least as large as that of y.

(ii) The binary relation ≧ is a continuous strictly monotonic weak order which satisfies comonotonic independence.

Furthermore the characteristic function v is uniquely determined. □

So, if the approach of subsection 4.1 applies, then ≧ satisfies the conditions mentioned in statement (ii) above, and reversedly, if ≧ satisfies the conditions in statement (ii) above, then *there exists* a characteristic function v such that by means of this the approach of subsection 4.1 applies. The implication (ii) ⇒ (i) above is mainly interesting in contexts where the characteristic function v is not easily available. As an example think of the case where players are ministers in a government, who during some years have been choosing among allocations of money over their departments. From their choices we can reveal "group preferences" of the form x ≧ y; if these preferences satisfy the conditions in statement (ii) above, then according to the above theorem we can derive from the choices of the ministers the characteristic function v. Then for any group S of ministers v(S) can be interpreted as an index for the power of this group of ministers.

5. THE RECIPE FOR THE ABOVE RESULTS, AND LITERATURE

The results presented in the previous sections were simple translations of results, formulated before in literature for decision making under uncertainty. The following translation has been involved everywhere :

Replace *state of nature* by *player*. (5.1)

5.1. *The translation algorithm of section 2*

In section 2 we translated results from a field of decision making under uncertainty which goes under the heading of "comparative probability theory". In comparative probability theory one considers a "more probable than" relation $\geqq$ on subsets (events) of the state space $\{1,\ldots,n\}$, and one searches for a probability measure agreeing in some way with the more-probable-than relation. So, besides the already mentioned translations, the following translations are involved :

Replace *event* by *coalition* (5.2)

Replace *S is more probable than T* by *the worth of S is higher than that of T* (5.3)

Replace *probability* by *allocation*. (5.4)

The condition of ordinal additivity has been introduced by de Finetti (1931). For a long time it was not known whether this condition would suffice, in presence of some "natural" presumptions, to guarantee the existence of an agreeing allocation/probability measure, see for instance Savage (1954, page 40/41). The matter was settled by Kraft, Pratt & Seidenberg (1959), who provided the Example 2.3, and gave the necessary and sufficient condition of (2.3). Their work used an algebraic notation which may not be easily accessible for every reader. Later Scott (1964) sketched a general procedure to use theorems of the alternative or separating hyperplane theorems to solve inequalities such as those involved in section 2, for finite state/player spaces. Since then, the conditions as in Theorem 2.4 are well-known. Jaffray (1974a,b) gave a more general approach by which also inequalities for infinite state/player spaces can be solved; by means of this technique Chateauneuf (1985) obtained necessary and sufficient conditions for the existence of an agreeing probability measure/allocation for general state/player spaces. The author of this paper studied the topic in Wakker (1979) and Wakker (1981), mainly for infinite state/player spaces. In Wakker (1981, Theorem 4) it was indicated that, with ordinal additivity presupposed, the characterization of almost agreement in Theorem 2.4 also holds for infinite state/player spaces. Also Gilboa (1985) considered questions of this nature; in his work a nonadditive characteristic function was interpreted as a distortion of an additive probability measure. A recent and complete overview of comparative probability theory is provided in Fishburn (1986).

5.2. *The translation algorithm of section 3*

The results of section 3 were obtained by translating work of Carnap on inductive reasoning, see Carnap (1962), Carnap & Jeffrey (1971), Fine (1973, section VII.D), Stegmüller (1973), Koerts & De Leede (1973), or Zabell (1981). As an example let us suppose that a die has been thrown several times. In this subsection 1,...,n are the sides of the die; after every throw exactly one side ("state of nature") will come up. Further : $d = (d_1,\dots,d_n)$ describes the number of times that the several sides have been observed after d_+ throws. And $f_j(d)$ designates the conditional subjective probability (Carnap preferred the interpretation as logical probability) that the $(d_+ + 1)$-th throw will give side j up, given the result of the previous throws. So the following translations are involved :

Replace *claim of player j* by *number of previously observed occurrences of side j of the die* (5.5)

Replace *proportion for player j* by *conditional probability for side j of the die* (5.6)

Like us, Carnap assumed monotonicity; so a new observation of a side of the die makes a next occurrence of this side more probable. And like us, Carnap assumed uninvolvedness of every side/player. Instead of our proportional uninvolvedness Carnap assumed "exchangeability", i.e. the probability of a sequence of outcomes depends only on the number of occurrences of the several sides of the die, and is independent of the particular order in which these sides occurred. This is equivalent to the equality, for all d,i,j :

$$f_i(d_1,\dots,d_i,\dots,d_j,\dots,d_n) \times f_j(d_1,\dots,d_i+1,\dots,d_j,\dots,d_n) =$$

$$f_j(d_1,\dots,d_i,\dots,d_j,\dots,d_n) \times f_i(d_1,\dots,d_i,\dots,d_j+1,\dots,d_n),$$

since the left-side gives the conditional probability that, given d, first side i will come up, next side j, whereas the right-side deals with the reversed order of occurrence of i and j.

In the presence of the other conditions, exchangeability is equivalent to proportional uninvolvedness of every pair i,j. Let us only show the derivation of exchangeability from proportional uninvolvedness, plus uninvolvedness. For any k such that $i \neq k \neq j$ (such a k exists since $n \geqq 3$)

$$\frac{f_i(d_1,\ldots,d_i,\ldots,d_j,\ldots,d_n)}{f_i(d_1,\ldots,d_i,\ldots,d_j+1,\ldots,d_n)} = \frac{f_k(d_1,\ldots,d_i,\ldots,d_j,\ldots,d_n)}{f_k(d_1,\ldots,d_i,\ldots,d_j+1,\ldots,d_n)} =$$

$$\frac{f_k(d_1,\ldots,d_i,\ldots,d_j,\ldots,d_n)}{f_k(d_1,\ldots,d_i+1,\ldots,d_j,\ldots,d_n)} = \frac{f_j(d_1,\ldots,d_i,\ldots,d_j,\ldots,d_n)}{f_j(d_1,\ldots,d_i+1,\ldots,d_j,\ldots,d_n)}$$

where the first equality follows from proportional uninvolvedness of i,k, the second from uninvolvedness of k, and the third from proportional uninvolvedness of j and k. The equality of the first and fourth quotient imply the equality given above as an equivalent of exchangeability.

Carnap showed that his conditions are equivalent to statement (i) in Theorem 3.4 (see for instance Zabell, 1981). This, together with the just derived observations, gives an alternative proof for our Theorem 3.4. The author studied Carnap's work for its applicability in probability calculations for the protection of statistical data files against anonimity disclosure, see Wakker (1986b).

5.3. *The translation algorithm of section 3*

The work of section 3 was obtained by translating work of Schmeidler for decision making under uncertainty, see Schmeidler (1984a,b,c). As an example, suppose a horse race will be held. There will participate n numbered horses, j is the "state of nature" that horse j will win. An *act* $x \in \mathbb{R}_+^n$ is a function from the states of nature to $\mathbb{R}_+$, interpreted as an investment (or bet, or whatever) that will result in a net gain of x_j if horse j will win. Now $\geqq$ denotes the preference relation of a decision maker over the set of acts, $x \geqq y$ meaning that the decision maker considers x to be at least as good as y. The characteristic function v is now interpreted as a nonadditive subjective probability measure for the decision maker; the higher v(S), where now S is an event, the more probable S is considered to be by the decision maker. So now the following translations are involved :

Replace *event* by *coalition* (5.7)

Replace *arbitrator* by *decision maker* (5.8)

Replace *allocation* by *act* (5.9)

Replace *characteristic function* by *subjective nonadditive probability* (5.10)

Schmeidler (1984a) showed the equivalence of statements (i) and (ii) in Theorem 4.2 in a slightly different context; in his work payment was not in money, as in section 4 above, but in lotteries over some set. The generalization of Schmeidler's work to the case where payment is in terms of elements of a "mixture space" (see for instance Wakker, 1986 a, Definition VII.2) is completely straightforward. One example of mixture spaces is the case of sets of lotteries over another set, as in Schmeidler's work; another example is $\mathbb{R}_+$, as in section 4 above. Hence in a mathematical sense Theorem 4.2 is completely analogous to Schmeidler (1984a, The Theorem). The author made use of Schmeidler's work on nonadditive probabilities in Wakker (1986a, Chapter VI).

6. CONCLUSION

This paper is based on the observation that the same mathematical structure is underlying many problems in decision making under uncertainty and in game theory. By simple translations, mainly by interchanging "state of nature" and "player", many results derived for decision making under uncertainty and game theory can be interchanged. This paper gave some examples. Admittedly, sometimes, such as in Definition 3.3, a minimal amount of creativity was needed. Still, an author in lack of inspiration, but in need of publications, may succeed with the following algorithm :

Take any theorems from a journal dealing with the topic of game theory, or probability theory/decision making under uncertainty.

Carry out the translations as described in this paper.

Send the resulting theorems to a journal dealing with the other topic than the original journal.

Do not refer to the original journal.

Do not refer to this paper.

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CHAPTER VII

THE CORE OF A COOPERATIVE GAME : BOUNDS AND CHARACTERIZATIONS

by Theo Driessen

1. A COOPERATIVE GAME IN CHARACTERISTIC FUNCTION FORM

This chapter is devoted to cooperative games in characteristic function form and the core concept for these games. In particular, we develop a theory concerning the relationship between the core and certain core catchers which are based on upper or lower bounds for the core. Necessary and sufficient conditions are given for the core catchers in question to coincide with the core.

The notion of a cooperative game in characteristic function form represents a mathematical model of a situation in which cooperation and side payments between the participants are allowed. In terms of costs, two or more participants in a multipurpose project can profit by cooperation because they can combine their plans with respect to common purposes in order to save expenses. Let $N = \{1,2,\dots,n\}$ be the set of all n participants who are supposed to cooperate in the undertaking of a joint project. For any nonempty subset S of participants, let $c(S)$ be the least cost of undertaking a similar joint venture which is designed solely for the purposes of the subset S. So, the above cooperative situation in which n participants are involved, generates a cost function $c: 2^N \to \mathbb{R}$ where the cost of the empty set $\emptyset$ is zero, i.e., we put $c(\emptyset) := 0$. Clearly, the various possibilities to meet the purposes of the union of two disjoint nonempty subsets S and T include the possibility to meet the purposes of S alone and T alone. As a consequence, the cost function $c: 2^N \to \mathbb{R}$ possesses the following property :

$$c(S \cup T) \leqq c(S) + c(T)$$
$$\text{for all } S,T \subset N \text{ with } S \cap T = \emptyset.$$

This property of the cost function c is known as the *subadditivity* of c. The cost savings which are due to the cooperation between all the participants instead of acting alone are given by $\sum_{j \in N} c(\{j\}) - c(N)$. The subadditivity of the cost function c guarantees that these cost savings of the set N of all participants are indeed nonnegative. Now the basic problem is to allocate these cost savings to the participants in an equitable and justifiable way. Usually, the proposed allocations of the total cost savings are to a greater or lesser extent based on the cost savings $\sum_{j \in S} c(\{j\}) - c(S)$ of all nonempty subsets S of participants. Thus, the above cooperative situation in terms of costs gives rise to a cost savings function $v: 2^N \to \mathbb{R}$ given by $v(\emptyset) := 0$ and

$$v(S) := \sum_{j \in S} c(\{j\}) - c(S) \qquad \text{for all } S \subset N,\ S \neq \emptyset.$$

The subadditivity of the cost function c implies the nonnegativity (i.e., $v(S) \geqq 0$ for all $S \subset N$) as well as the *superadditivity* of the associated cost savings function v, i.e.,

$$v(S \cup T) \geqq v(S) + v(T) \tag{1.1}$$
$$\text{for all } S,T \subset N \text{ with } S \cap T = \emptyset.$$

The superadditivity condition (1.1) expresses that it is advantageous (with respect to the cost savings) for any two disjoint subsets of participants to form their union. Moreover, the cost savings function v on 2^N is said to be zero-normalized because $v(\{i\}) = 0$ for all $i \in N$. The ordered pair $(N;v)$ itself is called a cooperative savings game.

DEFINITION. Let $n \in \mathbb{N}$ where $n \geqq 2$. A *cooperative n-person game in characteristic function form* is an ordered pair $(N;v)$ where $N := \{1,2,\ldots,n\}$ and $v: 2^N \to \mathbb{R}$ is a real-valued function on the set 2^N of all subsets of N.

Any subset S of the player set N (notation : $S \subset N$) is called a *coalition* and the *worth* $v(S)$ of the coalition S in the game $(N;v)$ represents the savings which can be achieved by cooperation solely between the members of the coalition S. For mathematical convenience, the savings of the empty coalition $\emptyset$ are zero, i.e., we always put $v(\emptyset) := 0$. In the general model, no other conditions are required to be satisfied by the so-called *characteristic function* v and hence, it may happen that the characteristic function v does not satisfy the superadditivity condition (1.1). As usual, we identify a cooperative game (N;v) with its characteristic function $v: 2^N \to \mathbb{R}$.

In order to elucidate the notion of a cooperative game in characteristic function form, we consider a production economy consisting of n traders who can increase the number of units of produced goods by cooperation. Various raw materials are needed to produce the goods concerned and each trader initially holds several units of the raw materials in question. The production process is supposed to be linear and each unit of a produced good can be sold at some net profit.
This economic situation can now be modelled as a cooperative n-person game (N;v) in characteristic function form where its player set N consists of the n traders and its characteristic function v is precisely the total net profit function. That is, the worth v(S) of a nonempty coalition S in the game (N;v) represents the largest possible monetary value of the goods produced by cooperation solely between the traders in the coalition S. Notice that the characteristic function v so defined must be superadditive.
The above cooperative game is known as the *linear production game* (Owen, 1975). We conclude this first section with three examples of linear production games. These three examples will also be used to illustrate the theory developed in the sequel.

EXAMPLE 1.1. Let the linear production economy consist of three traders and three raw materials where the i-th trader merely holds one unit of the i-th raw material M_i, $1 \leqq i \leqq 3$. One unit of the first (second respectively) good requires one unit of the first (second) and third raw material, whereas the second good G_2 is twice as valuable as the first good G_1. So, the production process has the form

$$\alpha M_1 + \alpha M_3 \to \alpha G_1 \qquad (\alpha \in \mathbb{N}),$$
$$\beta M_2 + \beta M_3 \to \beta G_2 \qquad (\beta \in \mathbb{N}).$$

Let $(N;v)$ be the associated linear production game and in addition, let $(N;w)$ be the linear production game generated by the above linear production economy in which the third trader holds two units (instead of one unit) of the third raw material. Then the 3-person games v and w are given by $N = \{1,2,3\}$ and

$$v(\{i\}) = 0 \quad \text{for all } i \in N, \qquad v(\{1,2\}) = 0,$$
$$v(\{1,3\}) = 1, \qquad v(\{2,3\}) = v(\{1,2,3\}) = 2,$$
$$w(\{1,2,3\}) = 3 \qquad \text{and } w(S) = v(S) \qquad \text{for all } S \subset N,\ S \neq N.$$

EXAMPLE 1.2. Let the linear production process with three raw materials M_1, M_2, M_3 and two goods G_1, G_2 be described by

$$\alpha M_1 + \alpha M_2 + \alpha M_3 \rightarrow \alpha G_1 \qquad (\alpha \in \mathbb{N}),$$
$$\beta M_1 + \beta M_2 + 2\beta M_3 \rightarrow \beta G_2 \qquad (\beta \in \mathbb{N}).$$

Further, the second good G_2 is twice as valuable as the first good G_1. The raw material bundles of the three traders are as follows :

(0,1,2), (1,0,1) and (1,1,0) respectively.

Then the associated linear production game v is given by

$$v(\{i\}) = 0 \qquad \text{for all } i \in N, \qquad v(\{2,3\}) = 1,$$
$$v(\{1,2\}) = v(\{1,3\}) = 2 \qquad \text{and } v(\{1,2,3\}) = 3.$$

EXAMPLE 1.3. Consider the linear production economy consisting of three traders and three raw materials M_1, M_2, M_3 which are needed in equal quantities to produce the unique good G. Thus, the production process is described by

$$\alpha M_1 + \alpha M_2 + \alpha M_3 \rightarrow \alpha G \qquad (\alpha \in \mathbb{N}).$$

A unit of the good G can be sold at a net profit of one unit of money. Further, the raw material bundles of the three traders are as follows :

(1,0,0), (1,2,1) and (1,1,2) respectively.

Then the associated linear production game w is given by

$$w(\{1\}) = 0, \qquad w(\{2\}) = w(\{3\}) = w(\{1,2\}) = w(\{1,3\}) = 1,$$
$$w(\{2,3\}) = 2 \qquad \text{and } w(\{1,2,3\}) = 3.$$

2. THE CORE OF A COOPERATIVE GAME

Since the introduction of the notion of a cooperative game in characteristic function form (Von Neumann and Morgenstern, 1944, page 240), many solution concepts for these games have been proposed. A solution concept prescribes for any game $(N;v)$ either no or at least one distribution of the total savings $v(N)$ of the grand coalition N among the players. A distribution of the amount $v(N)$ among the players in an n-person game $(N;v)$ is represented by an n-tuple $x = (x_1, x_2, \ldots, x_n)$ of real numbers satisfying the so-called *efficiency* condition $\sum_{j \in N} x_j = v(N)$. Here the i-th component x_i is interpreted as the monetary award to player $i \in N$ according to the allocation $x \in \mathbb{R}^n$. The total allocated award to the members of any nonempty coalition S is usually denoted by $x(S)$ instead of $\sum_{j \in S} x_j$ and further, we always put $x(\emptyset) := 0$.

The most well-known solution concept for cooperative games is the core which was first introduced and named in game theory in Gillies (1953). Informally, an efficient allocation belongs to the core of the given game if it is disadvantageous for the members of any nonempty coalition to withdraw from the allocation concerned in order to form their own coalition. In other words, the core consists of efficient allocations that can not be improved upon by any nonempty coalition.

DEFINITION. The *core* $C(v)$ and the *uppercore* $UC(v)$ of an n-person game $(N;v)$ are given by

$$\begin{aligned} C(v) &:= \{x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x(S) \geqq v(S) \text{ for all } S \subset N\}, \\ UC(v) &:= \{x \in \mathbb{R}^n \mid x(S) \geqq v(S) \text{ for all } S \subset N\}. \end{aligned} \tag{2.1}$$

The one-person constraints $x_i \geqq v(\{i\})$ for all $i \in N$ are known as the *individual rationality* condition for the allocation $x \in \mathbb{R}^n$ in the n-person game v. These one-person constraints express that no player $i \in N$ receives less than the alternate worth $v(\{i\})$ which he can attain for himself in the game v. An efficient allocation which also satisfies the individual rationality condition, is called an *imputation*. The set of all imputations for an n-person game $(N;v)$ is denoted by $I(v)$, i.e.,

$$I(v) := \{x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } x_i \geqq v(\{i\}) \text{ for all } i \in N\}.$$

The coalition constraint $x(S) \geqq v(S)$ is known as the *group rationality* condition for the allocation $x \in \mathbb{R}^n$ with respect to the nonempty coalition S in the n-person game v. The uppercore consists of (not necessarily efficient) allocations that satisfy all the group rationality conditions in the game. Obviously, the uppercore of an n-person game is a nonempty unbounded subset of $\mathbb{R}^n$. However, the 2^n-1 uppercore constraints in an n-person game may be inconsistent with the efficiency principle and hence, the core of a game may be empty. In the remainder of this section we pay attention to the conditions that determine whether or not the game has a nonempty core. As a matter of fact, a balancedness condition is necessary and sufficient for the game to possess a nonempty core.

DEFINITION. Let $N = \{1,2,\ldots,n\}$.
A *collection* $\mathcal{B} = \{S_1,S_2,\ldots,S_k\}$ of distinct nonempty coalitions is said to be *balanced* over N if there exist positive numbers $\alpha_1,\alpha_2,\ldots,\alpha_k$, such that

$$\sum_{j;i\in S_j} \alpha_j = 1 \qquad \text{for all } i \in N. \tag{2.2}$$

An n-person *game* $(N;v)$ is said to be *balanced* if for any balanced collection $\mathcal{B} = \{S_1,S_2,\ldots,S_k\}$ over N with corresponding positive numbers $\alpha_1,\alpha_2,\ldots,\alpha_k$, we have

$$\sum_{j=1}^{k} \alpha_j v(S_j) \leqq v(N). \tag{2.3}$$

Any player participates in several coalitions of the balanced collection $\mathcal{B}$ over the player set N and the proportions of these participations are described with the aid of the positive numbers corresponding to the coalitions of the balanced collection $\mathcal{B}$. The associated positive numbers can be regarded as weights for the coalitions of the balanced collection. Notice that the weights are equal to one if and only if the balanced collection over N is a partition of the set N. Hence, the notion of a balanced collection is a generalized version of the partitioning notion.

The balancedness condition (2.3) for a game expresses that it is not advantageous with respect to the savings in the game to divide the player set N into the coalitions of any balanced collection over N on the understanding that the savings of the coalitions are adjusted in accordance with the corresponding weights.
The next fundamental theorem concerning the core states that a game possesses a nonempty core if and only if the game is balanced. The result is due to Bondareva (1963, in Russian) as well as Shapley (1967).

THEOREM 2.1. The following two statements for an n-person game (N;v) are equivalent.
(i) The core of the game (N;v) is nonempty, i.e., $C(v) \neq \emptyset$.
(ii) The game (N;v) is balanced.

PROOF. For any coalition $S \subset N$ we first define the indicator function $1_S: N \to \{0,1\}$ by

$$1_S(i) = 1 \quad \text{if and only if } i \in S.$$

(a) We prove the implication (i) ⇒ (ii). Suppose $C(v) \neq \emptyset$ and let $\mathcal{B} = \{S_1, S_2, \ldots, S_k\}$ be a balanced collection over N with corresponding positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$. Because $C(v) \neq \emptyset$, there exists $x \in C(v)$. By (2.2) and (2.1) respectively, we have

$$\sum_{j=1}^{k} \alpha_j 1_{S_j}(i) = 1 \quad \text{for all } i \in N,$$

$$v(S_j) \leqq x(S_j) \quad \text{for all } 1 \leqq j \leqq k \qquad \text{and } x(N) = v(N).$$

Now it follows that

$$\sum_{j=1}^{k} \alpha_j v(S_j) \leqq \sum_{j=1}^{k} \alpha_j \sum_{i \in S_j} x_i = \sum_{j=1}^{k} \alpha_j \sum_{i \in N} x_i 1_{S_j}(i)$$

$$= \sum_{i \in N} x_i \sum_{j=1}^{k} \alpha_j 1_{S_j}(i) = \sum_{i \in N} x_i \cdot 1 = x(N) = v(N).$$

So, the balancedness condition (2.3) holds. This completes the proof of the implication (i) ⇒ (ii).

(b) Secondly, we prove the converse implication (ii) ⇒ (i). For that purpose, we fix some notation. Put $m := 2^n - 1$, let $\{S_1, S_2, \ldots, S_m\}$ be an arbitrary ordering of the set of all nonempty coalitions and further, let the vectors $b \in \mathbb{R}^n$, $c \in \mathbb{R}^m$ and the real-valued $m \times n$ matrix $A = [a_{ji}]$ be defined by

$$b_i := 1, \qquad c_j := v(S_j), \qquad a_{ji} := 1_{S_j}(i)$$

for all $1 \leqq i \leqq n$, $1 \leqq j \leqq m$. For any $x \in \mathbb{R}^n$ we have that $Ax \geqq c$ is equivalent to $x(S) \geqq v(S)$ for all $S \subset N$, $S \neq \emptyset$, and hence, $Ax \geqq c$ iff $x \in UC(v)$. From this and $UC(v) \neq \emptyset$, we obtain

$$\min\left[\sum_{i=1}^{n} b_i x_i \mid x \in \mathbb{R}^n,\ Ax \geqq c\right] = \min[x(N) \mid x \in UC(v)] \geqq v(N).$$

Due to this, the duality theorem for linear programs yields

$$\min\left[\sum_{i=1}^{n} b_i x_i \mid x \in \mathbb{R}^n,\ Ax \geqq c\right] = \max\left[\sum_{j=1}^{m} y_j c_j \mid y \in \mathbb{R}^m_+,\ yA = b\right].$$

Here $y \in \mathbb{R}^m_+$ means $y \in \mathbb{R}^m$ such that $y_j \geqq 0$ for all $1 \leqq j \leqq m$. Note that the vector equality $yA = b$ is equivalent to $\sum_{j=1}^{m} y_j 1_{S_j}(i) = 1$ for all $1 \leqq i \leqq n$. In view of this, the positive coordinates of a non-negative vector $y \in \mathbb{R}^m$ satisfying $yA = b$ can be regarded as the weights of the corresponding coalitions. In other words, any $y \in \mathbb{R}^m_+$ satisfying $yA = b$ can be associated in a natural way with a balanced collection over the player set N.

Suppose that (ii) holds. By the above reasoning, the balancedness of the game v implies

$$\max\left[\sum_{j=1}^{m} y_j c_j \mid y \in \mathbb{R}^m_+,\ yA = b\right] = \max\left[\sum_{j=1}^{m} y_j v(S_j) \mid y \in \mathbb{R}^m_+,\ yA = b\right] \leqq v(N).$$

Now it follows that $\min[x(N) \mid x \in UC(v)] = v(N)$ or equivalently, there exists $x \in UC(v)$ such that $x(N) = v(N)$. By (2.1), we get $x \in C(v)$ and in particular, $C(v) \neq \emptyset$. This completes the proof of the implication (ii) ⇒ (i). □

3. THE INVARIANCE OF THE CORE UNDER SUPERADDITIVE COVERING

It is desirable that the notion of the core behaves in a natural way with respect to plausible transformations on games. We first consider the game-theoretic version of the positive affine transformations known from geometry and calculus. Given an n-person game $(N;v)$, a positive real number α and an n-tuple $d = (d_1, d_2, \ldots, d_n)$ of real numbers, we define the n-person game $(N;\alpha v + d)$ by

$$(\alpha v + d)(S) := \alpha v(S) + \sum_{j \in S} d_j \qquad \text{for all } S \subset N.$$

In view of (2.1), it is straightforward to verify that the core behaves in a natural way with respect to these changes in scale, i.e., $C(\alpha v + d) = \alpha C(v) + d$. This invariance property of the core is known as the relative invariance under strategic equivalence.

Next we pay attention to a procedure which transforms a nonsuperadditive game into a closely relatedly superadditive game without affecting the core. The transformation in question is based on the idea of the least superadditive game that majorizes the original game.

DEFINITION. The *superadditive cover* $(N;\hat{v})$ of a game $(N;v)$ is given by $\hat{v}(\emptyset) := 0$ and

$$\hat{v}(S) := \max\Big[\sum_{j=1}^{k} v(S_j) \;\Big|\; \{S_1, S_2, \ldots, S_k\} \text{ is a partition of } S\Big]$$

$$\text{for all } S \subset N,\ S \neq \emptyset. \tag{3.1}$$

The worth $\hat{v}(S)$ of a nonempty coalition S in the game $(N;\hat{v})$ represents the largest possible total savings in the original game $(N;v)$ that is attained by dividing the members of S into pairwise disjoint coalitions. The term superadditive cover is explained by the following three properties of the game $\hat{v}$.

LEMMA 3.1. Let $(N;v)$ be a game.

(i) The game $(N;\hat{v})$ majorizes the game $(N;v)$, i.e.,
$\hat{v}(S) \geqq v(S)$ for all $S \subset N$.

(ii) The game $(N;\hat{v})$ is superadditive, i.e.,
$\hat{v}(S \cup T) \geqq \hat{v}(S) + \hat{v}(T)$ for all $S,T \subset N$ with $S \cap T = \emptyset$.

(iii) If $(N;w)$ is a superadditive game such that $w(S) \geqq v(S)$ for all $S \subset N$, then also $w(S) \geqq \hat{v}(S)$ for all $S \subset N$.

PROOF. (i) Let $S \subset N$, $S \neq \emptyset$. Because $\{S\}$ is the trivial partition of the nonempty set S, we derive immediately from formula (3.1) that the inequality $\hat{v}(S) \geqq v(S)$ holds.

(ii) Let $S,T \subset N$ be such that $S \cap T = \emptyset$. In case $S = \emptyset$ or $T = \emptyset$, then the equality holds in the superadditivity condition. So, suppose $S \neq \emptyset$ and $T \neq \emptyset$. By (3.1), there exist partitions $\{S_1,S_2,\ldots,S_k\}$ of S and $\{T_1,T_2,\ldots,T_m\}$ of T such that $\hat{v}(S) = \sum_{j=1}^{k} v(S_j)$ and $\hat{v}(T) = \sum_{j=1}^{m} v(T_j)$. Because $S \cap T = \emptyset$, we obtain that $\{S_1,S_2,\ldots,S_k,T_1,T_2,\ldots,T_m\}$ is a partition of $S \cup T$. Now it follows from (3.1) that

$$\hat{v}(S \cup T) \geqq \sum_{j=1}^{k} v(S_j) + \sum_{j=1}^{m} v(T_j) = \hat{v}(S) + \hat{v}(T).$$

(iii) Let $(N;w)$ be a superadditive game such that $v(T) \leqq w(T)$ for all $T \subset N$. We show that $\hat{v}(S) \leqq w(S)$ for all $S \subset N$. Let $S \subset N$, $S \neq \emptyset$, and let $\{S_1,S_2,\ldots,S_k\}$ be a partition of S. Then $v(S_j) \leqq w(S_j)$ for all $1 \leqq j \leqq k$ and hence, $\sum_{j=1}^{k} v(S_j) \leqq \sum_{j=1}^{k} w(S_j) \leqq w(S)$ where the last inequality follows from the superadditivity of the game w. From this and the formula (3.1), we conclude that $\hat{v}(S) \leqq w(S)$. □

From the above lemma we deduce that a game $(N;v)$ is superadditive if and only if $\hat{v}(S) = v(S)$ for all $S \subset N$. According to the next theorem, the cores of a balanced game and its superadditive cover are identical. The result is due to Aumann and Drèze (1974).

THEOREM 3.2. If a game $(N;v)$ possesses a nonempty core, then $C(\hat{v}) = C(v)$.

PROOF. Let $(N;v)$ be a game with $C(v) \neq \emptyset$. We assert that

$$x(S) \geqq \hat{v}(S) \text{ for all } x \in C(v) \text{ and all } S \subset N. \qquad (3.2)$$

(i) In order to prove the statement (3.2), let $x \in C(v)$ and $S \subset N$, $S \neq \emptyset$. Further, let $\{S_1, S_2, \ldots, S_k\}$ be a partition of S. Because $x \in C(v)$, we have that $v(S_j) \leqq x(S_j)$ for all $1 \leqq j \leqq k$. From this we deduce

$$\sum_{j=1}^{k} v(S_j) \leqq \sum_{j=1}^{k} x(S_j) = \sum_{j=1}^{k} \sum_{i \in S_j} x_i = \sum_{i \in S} x_i = x(S).$$

Together with the formula (3.1), this implies $\hat{v}(S) \leqq x(S)$. So, (3.2) holds.

(ii) Secondly, we assert $\hat{v}(N) = v(N)$. Choose $y \in C(v)$. Then (3.2) and (2.1) respectively yield $\hat{v}(N) \leqq y(N) = v(N)$. Moreover, $\hat{v}(N) \geqq v(N)$ by Lemma 3.1(i). Now it follows that the equality $\hat{v}(N) = v(N)$ holds.

(iii) The inclusion $C(v) \subset C(\hat{v})$ is a direct consequence of $v(N) = \hat{v}(N)$ and (3.2). In addition, the inverse inclusion $C(\hat{v}) \subset C(v)$ follows immediately from $\hat{v}(N) = v(N)$ and the fact that $\hat{v}(S) \geqq v(S)$ for all $S \subset N$ by Lemma 3.1(i). Therefore, $C(\hat{v}) = C(v)$ as was to be shown. □

The above theorem expresses that the core is invariant under superadditive covering on the class of balanced games. We conclude this section with some remarks concerning the invariance of the core under totally balanced covering on the class of balanced games.

The notion of total balancedness is closely related to the balancedness notion because a game $(N;v)$ is said to be *totally balanced* if the induced subgames $(T;v_T)$ are balanced for all $T \subset N$, $T \neq \emptyset$. Here the subgame $(T;v_T)$ with player set T is determined by the restriction of the characteristic function v to the set of all subsets of T, i.e., $v_T(S) := v(S)$ for all $S \subset T$. Due to Theorem 2.1, a game is totally balanced if and only if any induced subgame possesses a nonempty core.

The totally balanced cover $(N;\bar{v})$ of a game $(N;v)$ is given by $\bar{v}(\emptyset) := 0$ and for all $S \subset N$, $S \neq \emptyset$,

$$\bar{v}(S) := \max_{\mathcal{B}} \sum_{j=1}^{k} \alpha_j v(S_j). \qquad (3.3)$$

The maximum in formula (3.3) is taken over all balanced collections

$\mathcal{B} = \{S_1,S_2,\ldots,S_k\}$ of distinct nonempty subsets of the set S and where the corresponding positive numbers $\alpha_1,\alpha_2,\ldots,\alpha_k$, satisfy the condition (2.2) in which the player set N is replaced by the nonempty coalition S. The term totally balanced cover is explained by the fact that the game $(N;\bar{v})$ is the least totally balanced game that majorizes the original game (N;v). From this, it follows that a game (N;v) is totally balanced if and only if $\bar{v}(S) = v(S)$ for all $S \subset N$. According to a result in Shapley and Shubik (1969), the cores of a balanced game and its totally balanced cover coincide.

Finally, we remark that the superadditive cover of a game is much easier to compute than the totally balanced cover of the given game.

4. CORE CATCHERS BASED ON UPPER BOUNDS FOR THE CORE

The core of a cooperative game is bounded below by the individual rationality condition. Together with the efficiency condition, this implies that the core is also bounded above. Therefore, the core of an n-person game is a bounded subset of $\mathbb{R}^n$. The purpose of this section is to study a relationship between the structure of the core and a given upper bound for the core.

Informally, an allocation is called an upper bound for the core of a game if the payoff to any player according to a core-element is at most the payoff to the player by the allocation concerned.

DEFINITION. The set UB(v) of *upper bounds for the core* of an n-person game (N;v) is given by

$$UB(v) := \{y \in \mathbb{R}^n \mid x_i \leqq y_i \text{ for all } x \in C(v) \text{ and all } i \in N\}.$$

In view of (2.1), it is clear that $UB(v) \subset UC(v)$ whenever $C(v) \neq \emptyset$. In general, an upper bound for the core fails to meet the efficiency principle. Obviously, there are various possibilities to construct an efficient allocation from a given upper bound for the core. The most naive way is as follows : start off at the upper bound for the core and proceed by letting merely one component decrease till the efficiency principle is met. The procedure can be restarted at each of the n components of the given upper bound for the core and it yields n efficient allocations. According to the next theorem, the convex hull of the set of these obtained efficient allocations is a core catcher because it always includes the core. For any $i \in N$, let $e^i \in \mathbb{R}^n$ denote the i-th unit vector in $\mathbb{R}^n$. So, $e^i_i := 1$ and $e^i_j := 0$ for all $j \in N-\{i\}$.

THEOREM 4.1. Let $(N;v)$ be a game and $y \in UB(v)$. Then

$$C(v) \subset \text{conv } \{y - [y(N) - v(N)]e^i \mid i \in N\}.$$

PROOF. The inclusion is trivial if $C(v) = \emptyset$. So, suppose $C(v) \neq \emptyset$. Put $\alpha := y(N) - v(N)$. Because $y \in UB(v) \subset UC(v)$, we have $y(N) \geqq v(N)$ and thus, $\alpha \geqq 0$. We distinguish the two cases $\alpha = 0$ and $\alpha > 0$ respectively.
(i) Suppose $\alpha = 0$. Then we must show that the inclusion $C(v) \subset \{y\}$ holds. Let $x \in C(v)$. Since $y \in UB(v)$, we have that $x_i \leqq y_i$ for all $i \in N$. Together with $x(N) = v(N) = y(N)$, this implies that $x_i = y_i$ for all $i \in N$. Hence, $x = y$ for all $x \in C(v)$ and so, $C(v) \subset \{y\}$.
(ii) Suppose $\alpha > 0$. Let $x \in C(v)$. For any $i \in N$, define the real number β_i by $\beta_i := \alpha^{-1}(y_i - x_i)$. Then $\beta_i \geqq 0$ for all $i \in N$ because $\alpha > 0$ and $y \in UB(v)$. Further, we get

$$\beta(N) = \alpha^{-1}(y - x)(N) = \alpha^{-1}[y(N) - x(N)] = \alpha^{-1}[y(N) - v(N)] = 1.$$

The last equality follows from the definition of the real number α. Now we obtain

$$\sum_{i\in N} \beta_i(y - \alpha e^i) = \sum_{i\in N} \beta_i y - \sum_{i\in N} \alpha\beta_i e^i$$

$$= \beta(N)y - \sum_{i\in N} (y_i - x_i)e^i = y - (y - x) = x.$$

We conclude that $x = \sum_{i\in N} \beta_i(y - \alpha e^i)$ where $\beta_i \geqq 0$ for all $i \in N$ and $\beta(N) = 1$. In other words, the core-element x can be written as a convex combination of the efficient allocations $y - \alpha e^i$, $i \in N$, as was to be shown. □

The above theorem expresses that each upper bound for the core generates a core catcher. Next we direct our attention to the conditions that determine whether or not the core catcher in question coincides with the core. For that purpose, it is useful to introduce a notion which measures the gap between the worth of a given coalition and the total payoff to the members of the coalition by a given allocation.

DEFINITION. The *gap function* $g^v: 2^N \times \mathbb{R}^n \to \mathbb{R}$ of an n-person game (N;v) is given by

$$g^v(S,x) := \sum_{j \in S} x_j - v(S) = x(S) - v(S)$$

$$\text{for all } S \subset N \text{ and all } x \in \mathbb{R}^n. \qquad (4.1)$$

Notice that $g^v(\emptyset,x) = 0$ for all $x \in \mathbb{R}^n$. The expression $g^v(S,x)$ is called the gap of the coalition S with respect to the allocation x in the game v. In view of (2.1), the uppercore of a game consists of allocations that give rise only to nonnegative gaps. In particular, an allocation belongs to the core of a game if and only if the corresponding gaps are nonnegative in such a way that the gap of the grand coalition is equal to zero. The next theorem describes a relationship between the structure of the core and a property for the gap function with respect to an upper bound for the core.

THEOREM 4.2. Let (N;v) be a game and $y \in UB(v)$. Then the following two statements are equivalent.

(i) $C(v) = \text{conv}\,\{y - g^v(N,y)e^i \mid i \in N\}$.

(ii) $0 \leqq g^v(N,y) \leqq g^v(S,y)$ for all $S \subset N$, $S \neq \emptyset$. (4.2)

PROOF. For any $i \in N$, define the vector $x^i \in \mathbb{R}^n$ by $x^i := y - g^v(N,y)e^i$. Then Theorem 4.1 yields $C(v) \subset \text{conv}\,\{x^i \mid i \in N\}$.
Further, we always have that the core is a convex set. Now it follows that

$$C(v) = \text{conv}\,\{x^i \mid i \in N\} \text{ iff } x^i \in C(v) \text{ for all } i \in N.$$

(a) We first prove the implication (i) ⇒ (ii). Suppose that (i) holds and let $S \subset N$, $S \neq \emptyset$. Choose $i \in S$. Then $x^i \in C(v)$ because (i) holds. By $y \in UB(v)$, we have that $x^i_j \leqq y_j$ for all $j \in N$ which is equivalent to $g^v(N,y) \geqq 0$. From $x^i \in C(v)$ we also deduce $v(S) \leqq x^i(S) = y(S) - g^v(N,y)$ or equivalently, $g^v(N,y) \leqq g^v(S,y)$. So, the condition (4.2) holds. This completes the proof of the implication (i) $\Rightarrow$ (ii).

(b) Secondly, we prove the converse implication (ii) $\Rightarrow$ (i). Suppose that (4.2) holds. We show that $x^i \in C(v)$ for all $i \in N$. Let $i \in N$ and $S \subset N$, $S \neq \emptyset$. By using the formula (4.1), we obtain

$$x^i(N) = y(N) - g^v(N,y) = v(N),$$
$$x^i(S) = y(S) - g^v(N,y) = g^v(S,y) - g^v(N,y) + v(S) \geqq v(S) \quad \text{whenever } i \in S,$$
$$x^i(S) = y(S) = g^v(S,y) + v(S) \geqq v(S) \quad \text{whenever } i \notin S$$

where the inequalities follow from (4.2). In view of (2.1), we conclude that $x^i \in C(v)$. This completes the proof of the implication (ii) $\Rightarrow$ (i). □

The condition (4.2) for a game expresses that, with respect to the given upper bound for the core of the game, the gap of the grand coalition is minimal among the gaps of the nonempty coalitions. As usual, the gap function is also required to be nonnegative. According to Theorem 4.2, the condition (4.2) is necessary and sufficient for the core of the game to coincide with the core catcher generated by the given upper bound for the core of the game.

In the remainder of this section we look at upper bounds for the core that are based on the marginal contributions of the players. Here the marginal contribution $v(S) - v(S-\{i\})$ of player $i \in N$ to the coalition S in the game $(N;v)$ represents the increase or decrease of the savings in the game v whenever player i joins the coalition $S-\{i\}$. We are especially interested in the marginal contribution of any player to the grand coalition N as well as the largest marginal contribution of any player in the game.

DEFINITION. The *τ-vector* $t^v \in \mathbb{R}^n$ and the *m-vector* $m^v \in \mathbb{R}^n$ of an n-person game $(N;v)$ are given by

$$t_i^v := v(N) - v(N-\{i\}) \tag{4.3}$$

$$m_i^v := \max[v(S) - v(S-\{i\}) \mid S \subset N,\ i \in S] \tag{4.4}$$

for all $i \in N$.

PROPOSITION 4.3. Let $(N;v)$ be a game. Then

(i) $m^v \in UB(v)$, $t^v \in UB(v)$ and $t^{\hat{v}} \in UB(v)$.

(ii) $t_i^v \leq m_i^v$ for all $i \in N$.

$t_i^{\hat{v}} \leq t_i^v$ for all $i \in N$ whenever $C(v) \neq \emptyset$.

$t_i^{\hat{v}} = t_i^v$ for all $i \in N$ whenever v is superadditive.

(iii) $g^v(N-\{i\},t^v) = g^v(N,t^v)$ for all $i \in N$.

(iv) $g^v(S-\{i\},m^v) \leq g^v(S,m^v)$ for all $i \in N$ and all $S \subset N$ with $i \in S$.

PROOF. (i) Let $x \in C(v)$ and $i \in N$. By (4.3) and (2.1), we obtain

$t_i^v = v(N) - v(N-\{i\}) = x(N) - v(N-\{i\}) \geq x_i$. So, $t^v \in UB(v)$.

Moreover, $m_i^v \geq t_i^v \geq x_i$ where the first inequality is a direct consequence of the formulas (4.3) - (4.4). Hence, $m^v \in UB(v)$.

Because Theorem 3.2 yields $C(v) = C(\hat{v})$, we also have $x \in C(\hat{v})$. By applying the first result to the superadditive cover $\hat{v}$ instead of the game v itself, we get $t^{\hat{v}} \in UB(\hat{v})$. Together with $x \in C(\hat{v})$, this implies $t_i^{\hat{v}} \geq x_i$. Now we conclude that $t^{\hat{v}} \in UB(v)$.

(ii) In case the game v is superadditive, then $\hat{v}(S) = v(S)$ for all $S \subset N$ and hence, $t_i^{\hat{v}} = t_i^v$ for all $i \in N$. Suppose now $C(v) \neq \emptyset$. As was shown in part (ii) of the proof of Theorem 3.2, $C(v) \neq \emptyset$ yields $\hat{v}(N) = v(N)$. In view of the formula (4.3), we obtain that for all $i \in N$

$$t_i^{\hat{v}} = \hat{v}(N) - \hat{v}(N-\{i\}) = v(N) - \hat{v}(N-\{i\}) \leq v(N) - v(N-\{i\}) = t_i^v$$

where the inequality follows from Lemma 3.1(i).

(iii) It follows immediately from formulas (4.1) and (4.3) that for all $i \in N$

$$g^v(N-\{i\},t^v) = t^v(N-\{i\}) - v(N-\{i\}) = t^v(N) - v(N) = g^v(N,t^v).$$

(iv) Let $i \in N$ and $S \subset N$ be such that $i \in S$. Then we deduce from formula (4.4) that $m_i^v \geqq v(S) - v(S-\{i\})$. In view of (4.1), this implies

$$g^v(S,m^v) = m^v(S) - v(S) \geqq m^v(S-\{i\}) - v(S-\{i\}) = g^v(S-\{i\},m^v). \qquad \square$$

The m-vector is known as Milnor's upper bound (Milnor, 1952), while a detailed study of the τ-vector is presented in Driessen (1985). Part (i) of the above proposition states that the τ- and m-vector of the game v itself as well as the τ-vector of the superadditive cover $\hat{v}$ are indeed upper bounds for the core of the game v. Part (ii) describes that the core bounds by the τ- and m-vector of the game are not so sharp as the core bounds by the τ-vector of the superadditive cover. Part (iv) expresses the monotonicity of the gap function with respect to Milnor's upper bound, i.e., the gaps weakly increase as the coalition grows. In particular, the grand coalition generates the largest gap. As a consequence, condition (4.2) applied to Milnor's upper bound reduces to the requirement that the corresponding gap function is constant and nonnegative for all nonempty coalitions. As an example, we consider the 3-person game v given by

$v(\{i\}) = 0$ for all $i \in N$, $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = 2$ and $v(\{1,2,3\}) = 4$.

Then $m^v = (2,2,2)$ and $g^v(S,m^v) = 2$ for all $S \subset N$, $S \neq \emptyset$. So, the condition (4.2) holds with respect to Milnor's upper bound m^v for the core of the game v. Due to Theorem 4.2, we obtain

$C(v) = \text{conv}\,\{m^v - g^v(N,m^v)e^i \mid i \in N\}$ or equivalently,

$C(v) = \text{conv}\,\{(0,2,2), (2,0,2), (2,2,0)\}$.

Finally, we pay attention to the examples of the first section in order to elucidate the theory concerning the core catchers. By Proposition 4.3(ii), the τ-vectors of a linear production game and its superadditive cover coincide because the linear production games are always superadditive. We denote by $CC(t^v)$ and $CC(m^v)$ respectively the core catchers generated by the τ- and m-vector of the game v in question.

EXAMPLE 4.4. Consider again the 3-person games v and w of Example 1.1. The determination of their cores by (2.1) yields

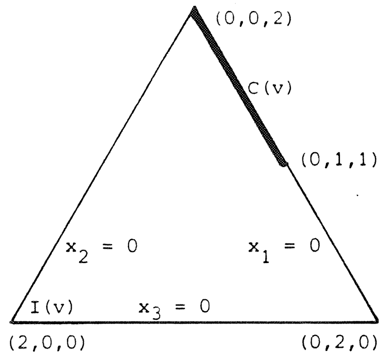

FIGURE 1a. The core C(v).

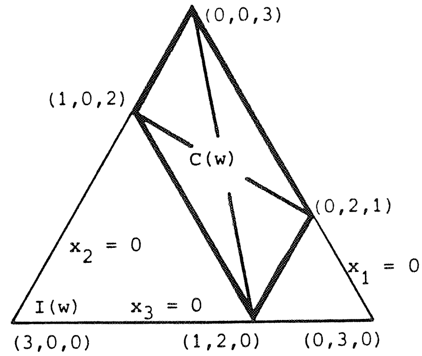

FIGURE 1b. The core C(w).

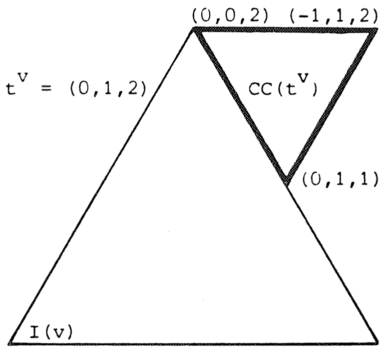

FIGURE 1c. The core catcher $CC(t^v)$.

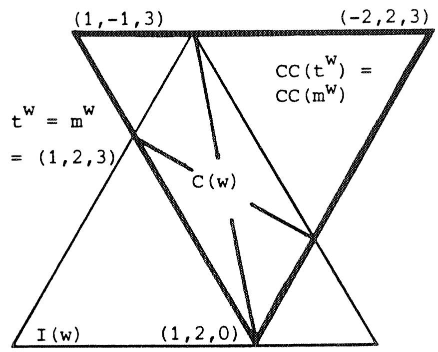

FIGURE 1d. The core catchers $CC(t^w)$ and $CC(m^w)$.

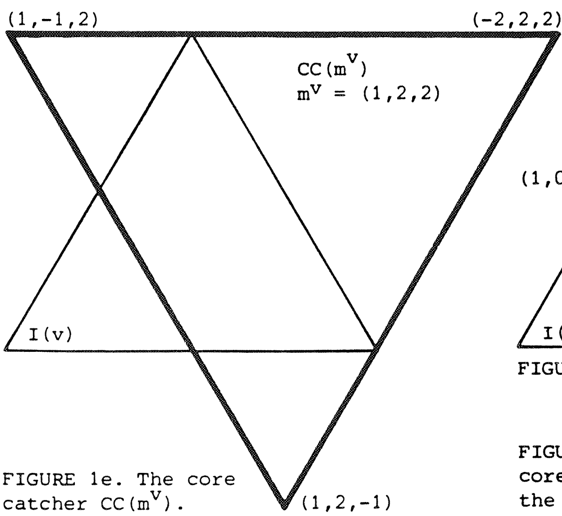

FIGURE 1e. The core catcher $CC(m^v)$.

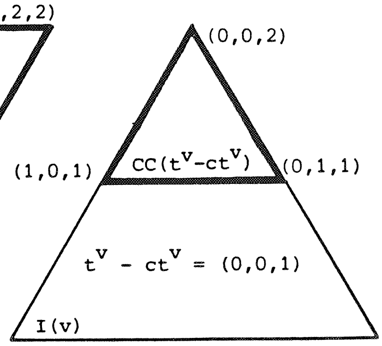

FIGURE 1f. The core catcher $CC(t^v-ct^v)$.

FIGURES 1a-f are related to the cores and the core catchers of the games v and w of Example 1.1.

$C(v) = \{x \in \mathbb{R}^3 \mid x_1 = 0,\ x_2 + x_3 = 2,\ 0 \leqq x_2 \leqq 1\}$ and
$C(w) = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 3,\ 0 \leqq x_1 \leqq 1,\ 0 \leqq x_2 \leqq 2\}$.

The core in each case is drawn in the Figures 1a-1b from which we deduce that the core of the game v is a straight line segment, whereas the core of the game w is a quadrilateral. In point of fact, the vertices of the core can be used to describe the entire core as follows :

$C(v) = \text{conv}\ \{(0,1,1),\ (0,0,2)\}$ and
$C(w) = \text{conv}\ \{(1,0,2),\ (1,2,0),\ (0,2,1),\ (0,0,3)\}$.

By formulas (4.3), (4.4) and (4.1) respectively, we obtain

$t^v = (0,1,2),\ m^v = (1,2,2),\ g^v(N,t^v) = 1,\ g^v(N,m^v) = 3,$
$t^w = (1,2,3) = m^w,\qquad g^w(N,t^w) = 3 = g^w(N,m^w).$

The gap functions of both games with respect to the τ- and m-vector are listed in Table 1. In the same table we observe that each of these four gap functions does not satisfy the corresponding condition (4.2). Therefore, the cores are strictly included in the core catchers generated by the τ- and m-vector. The geometric positions of the cores inside these core catchers are drawn in the Figures 1c-1e. Notice that $CC(m^w) = CC(t^w)$ because $m^w = t^w$.

coalition S	$v(S)$	$g^v(S,t^v)$	$g^v(S,m^v)$	$w(S)$	$g^w(S,t^w)$	$g^w(S,m^w)$
{1}	0	0	1	0	1	1
{2}	0	1	2	0	2	2
{3}	0	2	2	0	3	3
{1,2}	0	1	3	0	3	3
{1,3}	1	1	2	1	3	3
{2,3}	2	1	2	2	3	3
{1,2,3}	2	1	3	3	3	3

TABLE 1. The games v and w of Example 1.1 and their corresponding gap functions.

$v(S)$	$g^v(S,t^v)$	$g^v(S,m^v)$	$w(S)$	$g^w(S,t^w)$	$g^w(S,m^w)$	coalition S
0	2	2	0	1	1	{1}
0	1	2	1	1	1	{2}
0	1	2	1	1	1	{3}
2	1	2	1	2	2	{1,2}
2	1	2	1	2	2	{1,3}
1	1	3	2	2	2	{2,3}
3	1	3	3	2	2	{1,2,3}

TABLE 2. The games v and w of the Examples 1.2 - 1.3 and their corresponding gap functions.

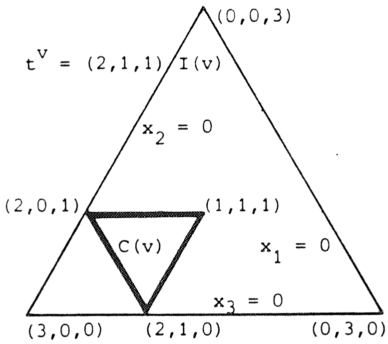

FIGURE 2a. The core C(v) and the core catcher $CC(t^V) = C(v)$.

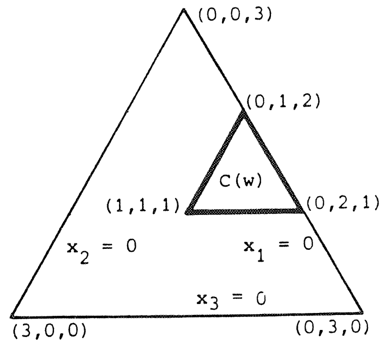

FIGURE 2b. The core C(w) satisfying C(w) = I(w).

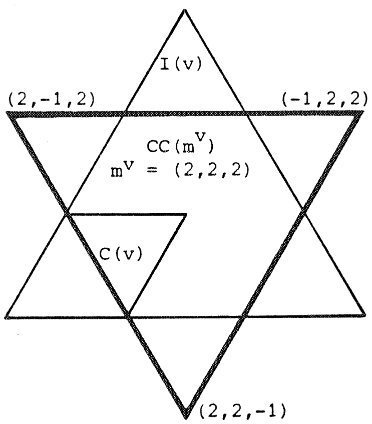

FIGURE 2c. The core catcher $CC(m^V)$.

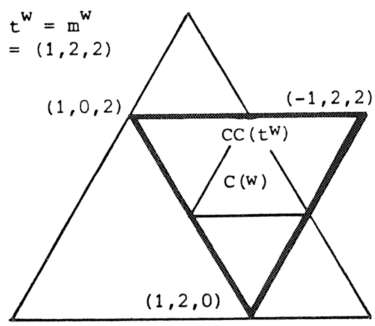

FIGURE 2d. The core catchers $CC(t^W)$ and $CC(m^W)$ satisfying $CC(t^W) = CC(m^W)$.

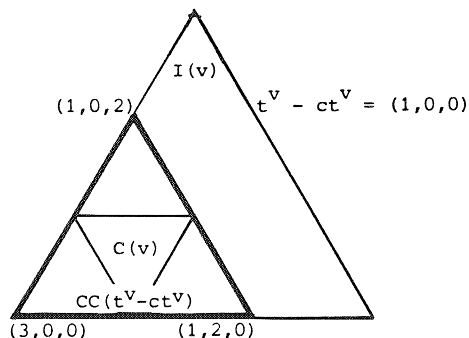

FIGURE 2e. The core catcher $CC(t^V-ct^V)$.

FIGURES 2a-e are related to the cores and the core catchers of the games v and w of the Examples 1.2.-1.3.

EXAMPLE 4.5. Consider again the 3-person games v and w of the Examples 1.2 and 1.3. The determination of their cores by (2.1) yields

$$C(v) = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 3,\ x_1 \leq 2,\ x_2 \leq 1,\ x_3 \leq 1\} \text{ and}$$
$$C(w) = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 3,\ x_1 \geq 0,\ x_2 \geq 1,\ x_3 \geq 1\} = I(w).$$

The core in each case is drawn in the Figures 2a-2b from which we deduce that both cores are equilateral triangles. Note that the core of the game w is precisely the imputation set of the game. As a matter of fact, the three vertices of the core can be used to describe the entire core as follows :

$$C(v) = \text{conv}\ \{(1,1,1),\ (2,0,1),\ (2,1,0)\} \text{ and}$$
$$C(w) = \text{conv}\ \{(1,1,1),\ (0,2,1),\ (0,1,2)\}.$$

By formulas (4.3), (4.4) and (4.1) respectively, we get

$$t^v = (2,1,1),\ m^v = (2,2,2),\ g^v(N,t^v) = 1,\ g^v(N,m^v) = 3,$$
$$t^w = (1,2,2) = m^w, \qquad g^w(N,t^w) = 2 = g^w(N,m^w).$$

The gap functions of both games with respect to the τ- and m-vector are listed in Table 2. In the same table we observe that the corresponding condition (4.2) is satisfied solely by the gap function of the game v with respect to the τ-vector t^v. Hence, the core of the game v coincides with the core catcher generated by the upper bound t^v for the core of the game v, i.e., $CC(t^v) = C(v)$. The geometric positions of the cores C(v) and C(w) respectively inside the core catchers $CC(m^v)$ and $CC(m^w)$, based on Milnor's upper bound for the core, are drawn in the Figures 2c-2d.

5. CORE CATCHERS BASED ON LOWER BOUNDS FOR THE CORE

The theory developed in the previous section is completely based on the notion of an upper bound for the core. Because the core is always bounded below by the individual rationality condition, it is also possible to look at lower bounds instead of upper bounds for the core. As was done in section 4 with respect to upper bounds, we now develop a similar theory concerning lower bounds for the core.

DEFINITION. The set LB(v) of *lower bounds for the core* of an n-person game (N;v) is given by

$$LB(v) := \{y \in \mathbb{R}^n \mid x_i \geqq y_i \text{ for all } x \in C(v) \text{ and all } i \in N\}.$$

Generally speaking, the efficiency principle is not met by a lower bound y for the core of a game v because the corresponding nonpositive gap $g^v(N,y)$ may fail to be zero. In order to construct an efficient allocation from a given lower bound for the core, we apply the following procedure : start off at the lower bound for the core and proceed by letting merely one component increase till the efficiency principle is met. The procedure can be restarted at each of the n components of the given lower bound y for the core of the n-person game v and it yields the efficient allocations $y - g^v(N,y)e^i$, $i \in N$. Part (i) of the next theorem states that the convex hull of the set of these obtained efficient allocations is a core catcher. Furthermore, in part (ii) of the theorem we formulate the condition which is necessary and sufficient for the core of the game to coincide with the core catcher generated by the given lower bound for the core of the game.

THEOREM 5.1. Let (N;v) be a game and $y \in LB(v)$. Then
(i) $C(v) \subset \text{conv}\{y - g^v(N,y)e^i \mid i \in N\}$.
(ii) $C(v) = \text{conv}\{y - g^v(N,y)e^i \mid i \in N\}$ if and only if

$$g^v(N,y) \leqq 0 \leqq g^v(S,y) \text{ for all } S \subset N,\ S \neq N. \tag{5.1}$$

The proof of the above theorem is completely similar to the proofs of the Theorems 4.1 and 4.2. Therefore, it is left to the reader to verify the validity of Theorem 5.1. By condition (5.1), the gap function with respect to the given lower bound for the core is required to be nonnegative except for the trivially nonpositive gap of the grand coalition.

As an example, we consider the lower bound for the core which is derived from the individual rationality condition. That is, the trivial lower bound $y \in \mathbb{R}^n$ for the core of an n-person game $(N;v)$ is determined by $y_i = v(\{i\})$ for all $i \in N$, whereas the associated efficient allocations $y - g^v(N,y)e^i$, $i \in N$, are precisely the vertices of the nonempty imputation set $I(v)$ for the game v. Hence, the core catcher generated by the trivial lower bound for the core is equal to the nonempty imputation set, i.e.,

$$\text{conv}\,\{y - g^v(N,y)e^i \mid i \in N\} = I(v) \tag{5.2}$$

whenever $I(v) \neq \emptyset$ and $y_i = v(\{i\})$ for all $i \in N$.

Usually, the core is a proper subset of the imputation set. From Theorem 5.1(ii) applied to the trivial lower bound for the core, we conclude that the core of a game $(N;v)$ coincides with the imputation set if and only if $v(N) \geqq \sum_{j \in N} v(\{j\})$ and

$$v(S) \leqq \sum_{j \in S} v(\{j\}) \text{ for all } S \subset N,\ S \neq N. \tag{5.3}$$

In case the game v is also superadditive, then (5.3) reduces to

$$v(S) = \sum_{j \in S} v(\{j\}) \text{ for all } S \subset N,\ S \neq N. \tag{5.4}$$

Condition (5.4) for a game expresses that the worth of any proper multi-person coalition is completely determined by the worths of the associated one-person coalitions. Notice that the linear production game of Example 1.3 satisfies condition (5.4) and as already observed in Example 4.5, its core does indeed coincide with the imputation set.

Next we present a procedure to construct a lower bound for the core from a given upper bound for the core. Let $y \in \mathbb{R}^n$ be an upper bound for the core of an n-person game $(N;v)$. Suppose that player $i \in N$ wants to participate in a multiperson coalition S and the other members of S are willing to cooperate with player i if they are paid for their cooperation according to the given upper bound y for the core. Player i receives the remaining part of the savings due to the formation of the coalition S. That is, as a reward for the participation in the coalition S, player i is allocated the amount

$v(S) - y(S-\{i\})$ or equivalently, $y_i - g^v(S,y)$.

In other words, player i receives less than y_i and the gap $g^v(S,y)$ represents the monetary value of the concession that player i has to make because of the participation in the coalition S. Subsequently, the largest possible payoff to player i is attained by minimization of the concession amounts $g^v(S,y)$ over all coalitions S containing player i.
In view of the above reasoning, we introduce a notion which describes the resulting concession amounts made by the single players with respect to an upper bound for the core.

DEFINITION. Let $(N;v)$ be an n-person game and $y \in UB(v)$. Then the associated *concession vector* $cy^v \in \mathbb{R}^n$ is given by

$$cy_i^v := \min[g^v(S,y) \mid S \subset N,\ i \in S] \tag{5.5}$$
for all $i \in N$.

PROPOSITION 5.2. Let $(N;v)$ be a game and $y \in UB(v)$.
(i) Then $y - cy^v \in LB(v)$.
(ii) In case $y = m^v$, then $y_i - cy_i^v = v(\{i\})$ for all $i \in N$.

PROOF. (i) Let $x \in C(v)$ and $i \in N$. We prove $cy_i^v \geqq y_i - x_i$. Let $S \subset N$ be such that $i \in S$. Because $x \in C(v)$ and $y \in UB(v)$, we have $x(S) \geqq v(S)$ and $y_j \geqq x_j$ for all $j \in S-\{i\}$. Now it follows that

$$g^v(S,y) = y(S) - v(S) \geqq y(S) - x(S) = (y - x)(S) \geqq y_i - x_i.$$

From this and formula (5.5), we conclude that $cy_i^v \geqq y_i - x_i$ or equivalently $x_i \geqq y_i - cy_i^v$. Hence, $y - cy^v \in LB(v)$.
(ii) Let $i \in N$. It follows directly from Proposition 4.3(iv) that $g^v(\{i\},m^v) \leqq g^v(S,m^v)$ for all $S \subset N$ with $i \in S$. Therefore, formula (5.5) applied to $y = m^v$ reduces to $cm_i^v = g^v(\{i\},m^v) = m_i^v - v(\{i\})$. □

Part (i) of the above proposition states that any upper bound for the core induces a lower bound for the core which is determined by the difference vector of the upper bound itself and the associated concession vector. In case Milnor's upper bound for the core is considered, then the induced lower bound for the core is precisely the trivial lower bound derived from the individual rationality condition. Due to (5.2), the core catcher generated by the lower bound $m^v - cm^v$ for the core coincides with the nonempty imputation set for the game v, i.e.,

$$CC(m^v - cm^v) = I(v) \text{ for all games } v \text{ with } I(v) \neq \emptyset.$$

Finally, we illustrate the developed theory by the examples of the first section.

EXAMPLE 5.3. Consider once again the 3-person games v and w of Examples 1.1 and 4.4. By using formula (5.5) and Table 1, we obtain $ct^v = (0,1,1)$, $t^v - ct^v = (0,0,1)$ and $g^v(N,t^v-ct^v) = -1$. Hence, the core catcher $CC(t^v-ct^v)$ is given by

$$CC(t^v-ct^v) = \text{conv}\{(1,0,1), (0,1,1), (0,0,2)\}.$$

The core catcher $CC(t^v-ct^v)$ is drawn in Figure 1f. Obviously, we have $C(v) \neq CC(t^v-ct^v)$ which is due to the fact that condition (5.1) is not satisfied by the gap function of the game v with respect to the lower bound $t^v - ct^v$ for the core, e.g., $g^v(\{2,3\},t^v-ct^v) = -1 < 0$. As usual, the core catcher $CC(m^v-cm^v) = I(v)$ and further, the equality $t^w = m^w$ implies $CC(t^w-ct^w) = CC(m^w-cm^w) = I(w)$.

EXAMPLE 5.4. Consider once again the 3-person games v and w of Examples 1.2, 1.3 and 4.5. By using Table 2 and formula (5.5), we get

$$ct^v = (1,1,1),\ t^v - ct^v = (1,0,0),\ g^v(N,t^v-ct^v) = -2,$$
$$ct^w = (1,1,1),\ t^w - ct^w = (0,1,1),\ g^w(N,t^w-ct^w) = -1.$$

Hence, $CC(t^v-ct^v) = \text{conv}\{(3,0,0), (1,2,0), (1,0,2)\}$ and the geometric position of the core C(v) inside the core catcher $CC(t^v-ct^v)$ is drawn in Figure 2e. The strict inclusion $C(v) \subset CC(t^v-ct^v)$ is due to the negative gaps $g^v(S,t^v-ct^v)$ for all two-person coalitions S. As usual, $CC(m^v-cm^v) = I(v)$, while $t^w = m^w$ implies $CC(t^w-ct^w) = CC(m^w - cm^w) = I(w)$. Nevertheless, the core $C(w)$ of the game w coincides with the core catchers concerned because condition (5.1) holds, i.e.,

$$g^w(N,t^w-ct^w) = -1 \text{ and}$$
$$g^w(S,t^w-ct^w) = 0 \text{ for all } S \subset N,\ S \neq N.$$

In point of fact, $C(w) = I(w)$ as already observed in Example 4.5.

6. CONCLUDING REMARKS

The notion of a core catcher is often used to characterize a specific subclass of games. The core catchers as studied in Theorems 4.1 and 5.1(i) possess a very regular structure. In fact, the equidistant n vertices of those core catchers are derived in a natural way from an upper or a lower bound for the core of an n-person game. It may happen that these n vertices of the core catcher in question degenerate into a single point which coincides with the given upper or lower bound for the core. Weber (1978) and Driessen (1985, cf. page 117) presented other core catchers which are useful to characterize the so-called convex and k-convex games respectively in terms of the core instead of the characteristic function itself. Here a game $(N;v)$ is said to be 1-convex if the corresponding gap function with respect to the τ-vector t^v satisfies condition (4.2), i.e., the game $(N;v)$ is 1-convex if and only if

$$0 \leqq g^v(N,t^v) \leqq g^v(S,t^v) \text{ for all } S \subset N,\ S \neq \emptyset. \tag{6.1}$$

We conclude this chapter with the detailed remark that the notion of the so-called nucleolus is closely related to the centre of gravity of the vertices of the core catcher generated by the τ-vector. The nucleolus of a game was introduced in Schmeidler (1969) and this one-point solution concept occupies a central position within the core of a balanced game. An interrelationship between the nucleolus $\eta(v) \in \mathbb{R}^n$ of an n-person game $(N;v)$ and the centre $t^v - n^{-1}g^v(N,t^v)\ (1,1,\dots,1)$ of the core catcher $CC(t^v)$ can be described in terms of the smallest marginal contribution of any player to the nonempty coalitions in the game.

DEFINITION. The *$\bar{m}$-vector* $\bar{m}^v \in \mathbb{R}^n$ of an n-person game (N;v) is given by

$$\bar{m}_i^v := \min[v(S) - v(S-\{i\}) \mid S \subset N,\ i \in S,\ S \neq \{i\}] \qquad (6.2)$$

for all $i \in N$.

The $\bar{m}$-vector is neither an upper bound nor a lower bound for the core. For instance, for the 3-person game w of Example 1.1 we have $\bar{m}^w = (0,0,1)$, whereas $(1,2,0) \in C(w)$. We mention two interesting results concerning the $\bar{m}$-vector.

THEOREM 6.1. Let (N;v) be a superadditive n-person game.

(i) (Kikuta, 1983). If the game (N;v) is such that $\bar{m}^v(N) > v(N)$, then the game (N;v) is totally balanced.

(ii) (Funaki, 1986). The following two statements are equivalent.

(1) $t_i^v - n^{-1}g^v(N,t^v) \leq \bar{m}_i^v$ for all $i \in N$. (6.3)

(2) $\eta_i(v) \leq \bar{m}_i^v$ for all $i \in N$.

In addition, $\eta(v) = t^v - n^{-1}g^v(N,t^v)(1,1,\ldots,1)$ if (6.3) holds.

(iii) (Driessen, 1985). $\eta(v) = t^v - n^{-1}g^v(N,t^v)(1,1,\ldots,1)$ if (6.1) holds.

Part (ii) of the above theorem states that the $\bar{m}$-vector $\bar{m}^v$ is an upper bound for the nucleolus $\eta(v)$ of a superadditive game v if and only if the $\bar{m}$-vector is an upper bound for the centre of the core catcher generated by the τ-vector t^v. In view of the formulas (6.2) and (4.1), the condition (6.3) for a game (N;v) is equivalent to

$$g^v(S,t^v) - g^v(S-\{i\},t^v) \leq n^{-1}g^v(N,t^v) \qquad (6.4)$$

for all $i \in N$ and all $S \subset N$ with $i \in S$, $S \neq \{i\}$.

From the Tables 1-2 we deduce that condition (6.4) is satisfied solely by the gap function of the 3-person game v of Example 1.2 and thus, its nucleolus $\eta(v) = t^v - n^{-1}g^v(N,t^v)(1,1,1) = \frac{1}{3}(5,2,2)$. Conditions (6.1) and (6.3) respectively are sufficient, but not necessary for the coincidence in question. For example, the 3-person game w of Example 1.3 does not satisfy conditions (6.1) and (6.3), but nevertheless its nucleolus $\eta(w) = \frac{1}{3}(1,4,4)$ coincides with the centre of the core catcher generated by the τ-vector t^w of the game w.

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CHAPTER VIII

THE τ-VALUE : A SURVEY

By Theo Driessen

1. INTRODUCTION

This chapter deals with a specific one-point solution concept for cooperative games in characteristic function form. The concept in question is called the τ-value and was introduced in Tijs (1981). The τ-value is based on the idea of an efficient compromise between suitably chosen upper and lower bounds for the core. The relevant bounds for the core were already considered in the previous chapter. Throughout this chapter, it is supposed that the reader is familiar with the notions treated in the previous chapter. In section 2 we present the notion of the τ-value. In section 3 we review various interesting results concerning the τ-value concept. In particular, an axiomatization of the τ-value will be mentioned.

2. THE τ-VALUE OF A QUASIBALANCED GAME

Let (N;v) be a balanced n-person game. Then the τ-vector $t^v \in \mathbb{R}^n$ of the game v is an upper bound for the core of the game v. As an initial step, we allocate the upper payoff t_i^v to player i, $i \in N$. Generally speaking, the efficiency principle is not met by the upper bound t^v for the core because the corresponding nonnegative gap $g^v(N,t^v)$ may fail to be zero. In the second and final step, each player is charged a part of the joint concession amount $g^v(N,t^v)$. Here the joint concession amount is prorated in proportion to the concession amounts made by the single players and described by means of the concession vector $ct^v \in \mathbb{R}^n$. The resulting efficient allocation is called the τ-value of the given balanced game.
An equivalent interpretation of the τ-value for a balanced game (N;v) is as follows. The upper bound t^v for the core of the game v induces the lower bound $t^v - ct^v$ for the core where the vector ct^v denotes the concession vector with respect to the upper bound t^v. We suppose that the players in

the game v regard their payoffs by the upper bound t^v as maximal payoffs and further, payoffs by the lower bound $t^v - ct^v$ are seen as minimal payoffs to the players. In other words, the relevant upper (lower respectively) bound for the core of the game is interpreted as the utopia (disagreement) allocation. The τ-value of the balanced game is obtained as an efficient compromise between the utopia and disagreement allocation. That is, the τ-value is obtained as the unique efficient allocation lying on the straight line segment with end points the utopia vector t^v and the disagreement vector $t^v - ct^v$. Due to this interpretation, the τ-value allocation is well-defined for games (N;v) satisfying

$$ct_i^v \geqq 0 \quad \text{for all } i \in N \quad \text{and} \quad (t^v - ct^v)(N) \leqq v(N) \leqq t^v(N)$$

or equivalently,

$$g^v(S,t^v) \geqq 0 \text{ for all } S \subset N \quad \text{and} \quad ct^v(N) \geqq g^v(N,t^v). \tag{1}$$

Condition (1) for a game (N;v) requires that the gap function of the game v with respect to the τ-vector t^v is nonnegative and in addition, the joint concession amount $g^v(N,t^v)$ is at most the total sum of the concession amounts made by the single players. Condition (1) is known as the *quasibalancedness* condition for the game (N;v). It is clear that balancedness implies quasibalancedness. The notion of the τ-value on the class of quasibalanced games is due to Tijs (1981).

DEFINITION. The *τ-value* $\tau(v) \in \mathbb{R}^n$ *of a quasibalanced n-person game* (N;v) is given by

$$\begin{aligned} \tau(v) &:= t^v && \text{if } g^v(N,t^v) = 0 \\ &= t^v - g^v(N,t^v)[ct^v(N)]^{-1}ct^v && \text{if } g^v(N,t^v) > 0. \end{aligned}$$

An extension of the τ-value from the class of quasibalanced n-person games to the class of n-person games with a nonempty imputation set is presented in Driessen (1985). The τ-value of a quasibalanced game is drawn in Figure 1.

As an example, we determine the τ-value of the balanced games considered in Examples 1.1 - 1.3 of the previous chapter. The relevant results are as follows.

game . of Example	τ-vector $t^{\cdot}$	concession vector $ct^{\cdot}$	gap $g^{\cdot}(N,t^{\cdot})$	τ-value $\tau(.)$
v of 1.1	(0,1,2)	(0,1,1)	1	$\frac{1}{2}(0,1,3)$
w of 1.1	(1,2,3)	(1,2,3)	3	$\frac{1}{2}(1,2,3)$
v of 1.2	(2,1,1)	(1,1,1)	1	$\frac{1}{3}(5,2,2)$
w of 1.3	(1,2,2)	(1,1,1)	2	$\frac{1}{3}(1,4,4)$

3. RESULTS CONCERNING THE τ-VALUE

First of all, we list various useful properties of the τ-value on the class QB^n of quasibalanced n-person games.

THEOREM. The τ-value $\tau : QB^n \to \mathbb{R}^n$ possesses the following six properties.

(i) *Efficiency*. For all $(N;v) \in QB^n$: $\sum_{j\in N} \tau_j(v) = v(N)$.

(ii) *Individual rationality*.
For all $(N;v) \in QB^n$ and all $i \in N$: $\tau_i(v) \geqq v(\{i\})$.

(iii) *Core stability*.
For all $(N;v) \in QB^n$ with $g^v(N,t^v) = 0$: $C(v) = \{\tau(v)\} = \{t^v\}$.

(iv) *Core stability*. For all $(N;v) \in QB^n$ with $g^v(N,t^v) > 0$:
$\tau(v) \in C(v)$ if and only if
$ct^v(N)[g^v(N,t^v)]^{-1} \geqq ct^v(S)[g^v(S,t^v)]^{-1}$
for all $S \subset N$ with $2 \leqq |S| \leqq n-2$ and $g^v(S,t^v) > 0$.

(v) *Core stability*.
For all $(N;v) \in QB^n$ where $2 \leqq n \leqq 3$: $\tau(v) \in C(v)$.

(vi) *Relative invariance under strategic equivalence*.
For all $(N;v) \in QB^n$, all $\alpha \in (0,\infty)$ and all $d \in \mathbb{R}^n$:
$\tau(\alpha v + d) = \alpha\tau(v) + d$.

Although the τ-value of a quasibalanced game is efficient as well as individually rational, it does not necessarily belong to the core of the game whenever there are at least four players. The τ-value of quasibalanced 3-person games is always included in the core. Note that the τ-value of the balanced 3-person games, considered in Examples 1.1 - 1.3 of the previous chapter, is even the centre of gravity of the vertices of the core. Part (vi) of the above theorem states that the τ-value behaves in a natural way with respect to changes in scale. Further, it is obvious from the definition that the τ-value of a quasibalanced game v is proportional to the τ-vector t^v of the game v whenever the concession vector ct^v is equal to the τ-vector t^v. According to the next theorem, this property together with the relative invariance under strategic equivalence fully characterize the τ-value on the class of quasibalanced n-person games. This axiomatization of the τ-value on the class QB^n is due to Tijs (1987).

THEOREM. The τ-value $\tau : QB^n \to \mathbb{R}^n$ is the unique function $\psi : QB^n \to \mathbb{R}^n$ with the following two properties :

(i) For all $(N;v) \in QB^n$, all $\alpha \in (0,\infty)$ and all $d \in \mathbb{R}^n$:
$\psi(\alpha v + d) = \alpha\psi(v) + d$.

(ii) For all $(N;v) \in QB^n$ with $ct^v = t^v$, the vector $\psi(v)$ is proportional to the τ-vector t^v of the game v.

In spite of the fact that the τ-value of a balanced game can be interpreted as an efficient compromise between an upper and a lower bound for the core of the game, it may fall outside the core. Nevertheless, the τ-value on the class of so-called 1-convex games occupies a central position within the core because it coincides with the centre of gravity of the vertices of the core.

THEOREM. Let $(N;v)$ be a *1-convex n-person game*, i.e.,

$$0 \leqq g^v(N,t^v) \leqq g^v(S,t^v) \qquad \text{for all } S \subset N,\ S \neq \emptyset.$$

Then $C(v) = \text{conv}\,\{t^v - g^v(N,t^v)e^i \mid i \in N\}$

and $\tau(v) = t^v - n^{-1}g^v(N,t^v)(1,1,\dots,1) \in C(v)$.

In addition, the nucleolus $\eta(v)$ equals the τ-value $\tau(v)$.

PROOF. The statement concerning the core structure is a direct consequence of Theorem 4.2 of the previous chapter applied to $y = t^v$. The 1-convexity of the game v implies that the concession vector ct^v is given by $ct_i^v = g^v(N,t^v)$ for all $i \in N$. In view of (1), it is clear that the game v is quasibalanced. Now the formula for the τ-value follows immediately from its definition and the τ-value belongs to the core because it represents the centre of gravity of the core. The statement concerning the nucleolus is valid because of Theorem 6.1(iii) of the previous chapter. □

Note that the game v of Example 1.2 of the previous chapter is 1-convex. The notion of 1-convexity was introduced in Driessen (1985) as an adjunct to the detailed study of the τ-value. There other results for the τ-value can be found.

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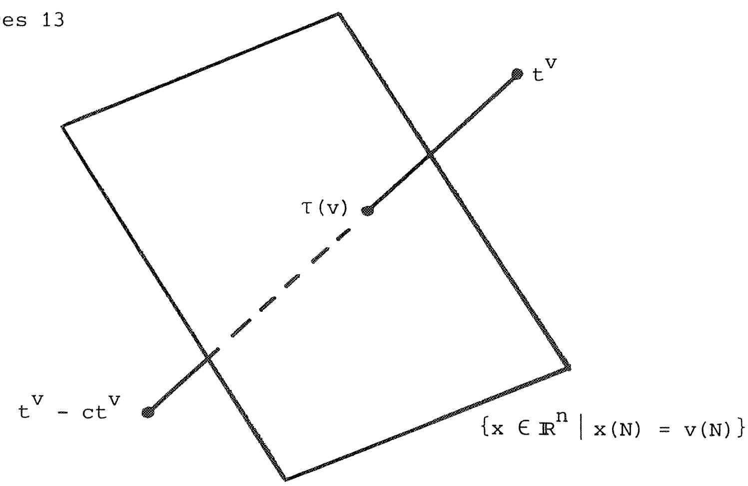

FIGURE 1. The τ-value τ(v) of a quasibalanced n-person game v.

CHAPTER IX

ON THE EXTREME ELEMENTS OF THE CLASS OF (0,1)-NORMALIZED SUPERADDITIVE GAMES

by Jean Derks

1. INTRODUCTION

From the introduction of the theory of cooperative games (in characteristic function form) in Von Neumann and Morgenstern (1944) the superadditive games have played a prominent role. Economical models which give rise to cooperative games usually impose the superadditivity property. Furthermore, the class of superadditive games, seen as a set in the game space, has a very nice structure. After a normalization procedure this class may be seen as a (bounded) polyhedron, i.e. an intersection of a finite system of closed halfspaces. From this point of view the characteristics of the class of (normalized) superadditive games will be known when its extreme elements are characterized.

Unfortunately the characterization of the extreme superadditive games is still an open problem. Only partial characterizations have been given such as in Gurk (1959) and Griesmer (1959). The main result in this work is the introduction of a new partial characterization. For this introduction we will use established concepts like "essential coalitions" and "decomposable games".

In the next section we will present the basic definitions from the theory of cooperative games. Section 3 is devoted to a survey of the results on extreme superadditive games which are known in literature. In section 4 the notion of uniform games is introduced. It is shown that the extremality of a uniform superadditive game is equivalent to the non-decomposability of the game. For the proof of this characterization a result is needed which has nothing in common with our main subject and, therefore, it is stated and proved in an appendix.

2. PRELIMINARIES

Let $N = \{1,2,\ldots,n\}$ denote a finite set of *players*. An *n-person cooperative game* v (in characteristic function form) is a real valued function on the set 2^N of subsets of N such that $v(\emptyset) = 0$. The subsets of N are called *coalitions*. A game v assigns to each coalition $S \subseteq N$ the *value* $v(S)$ which may be seen as the maximal gain of the coalition S which can be reached by the players in S when they join their resources.
If a game v fullfills

$$v(S) + v(N\backslash S) = v(N) \text{ for each coalition } S \subseteq N$$

then v is said to be a *constant sum* game. In this chapter we will specially pay attention to games v which fullfill

$$v(S) + v(T) \leqq v(S \cup T) \text{ for all disjoint coalitions } S,T \subseteq N.$$

Such games are called *superadditive*. In superadditive games it is not unprofitable for two disjoint coalitions to cooperate. For a more detailed description of the notion of a cooperative games and examples of superadditive games the reader is reffered to chapter 6.
An other property which will be mentioned frequently is the convexity property. A game v is said to be *convex* if

$$v(S) + v(T) \leqq v(S \cup T) + v(S \cap T) \text{ for all coalitions } S,T \subseteq N.$$

One immediately sees that for a superadditive game v we have

$$v(S) \geqq \Sigma_{i:i\in S}\, v(\{i\}) \quad \text{for each coalition } S \subset N. \tag{1}$$

Let α_v denote the constant $v(N) - \Sigma_{i:i\in N}\, v(\{i\})$.
Whenever α_v is positive we define the game v_0 as follows

$$v_0(S) = \alpha_v^{-1}\, (v(S) - \Sigma_{i:i\in S}\, v(\{i\})) \quad \text{for each } S \subseteq N.$$

Because of (1) it is clear that v_0 assigns a non-negative value to each coalition. The game v_0 is called the *(0,1)-normalization* of v and it assigns 0 to the one-person coalitions and 1 to the grand coalition N. Such games are called *(0,1)-normalized*.
This normalization procedure is a well-known concept in the cooperative game theory. We will often apply this normalization. Specially we will use the fact that the values of the coalitions in a (0,1)-normalized superadditive game v are restricted between 0 and 1 :

$$0 \leq v(S) \leq v(S) + v(N\setminus S) \leq v(N) = 1 \text{ for each coalition } S \subseteq N. \quad (2)$$

Examples of (0,1)-normalized games are the so-called unanimity games u_T, with $T \in 2^N \setminus \{\emptyset,\{1\},\{2\},\ldots,\{n\}\}$, defined as

$$u_T(S) = \begin{cases} 1 \text{ if } S \supseteq T, \\ 0 \text{ otherwise} \end{cases} \quad \text{for each } S \subseteq N.$$

Note that the unanimity games are convex and, thus, superadditive. Furthermore, the values for the coalitions are only 0 and 1. Such games are called *simple* games.

Let K_n denote the set of (0,1)-normalized superadditive n-person games. From (2) we conclude that the set K_n is a bounded set in the game space and the reader is invited to check that the set K_n is convex, i.e. for each two elements v and w of K_n and scalar α, $0 < \alpha < 1$, we have

$$\alpha v + (1-\alpha)w \in K_n,$$

where $\alpha v + (1-\alpha)w$ is defined to be the game which assigns to each coalition S the value $\alpha v(S) + (1-\alpha)w(S)$. The game $\alpha v + (1-\alpha)w$ is called a convex combination of v and w.

A game $v \in K_n$ is called *extreme* if it is not a convex combination of two other games of K_n, i.e. if $v = \alpha w_1 + (1-\alpha)w_2$ for two games $w_1, w_2 \in K_n$ and a scalar α, $0 < \alpha < 1$, then $v = w_1 = w_2$.

Also the set $L_n = \{v \in K_n : v \text{ a constant sum game}\}$ is a convex set. Its extreme elements are, as for K_n, the games in L_n which cannot be written as a convex combinations of two different games of L_n.

In the following section we will give a summary of the results concerning the extreme games of K_n and L_n which can be found in literature. One proof is included as an example of the argumentation techniques which are used frequently.

3. A SURVEY OF RESULTS IN LITERATURE

In Von Neumann and Morgenstern (1944) the notion of a cooperative game in characteristic function form is introduced. There also the study on the subject of extreme superadditive games started. The game theoretical model which gave rise to the definition of a cooperative game imposed the superadditivity and the constant sum property on the game. Therefore, the first results concern the set L_n and Von Neumann and Morgenstern showed that the extreme games of L_4 are exactly the simple games of L_4.

The question raised whether this was generally true. A negative answer was given in Gurk (1959) by characterizing the extreme games of L_5. First Gurk noticed that each simple game in L_n is extreme. Then the considered *three-valued* games, i.e. games in which the value of each coalition can admit only three values. For (0,1)-normalized constant sum games these values are, of course, 0, $\frac{1}{2}$ and 1. Furthermore the following notion is introduced :

Let v be an element of L_n.
A sequence $(T_1,\ldots,T_{2R})$ of coalitions is called a *K-chain* if

i) $T_1 = T_{2R}$,

ii) $T_r \cap T_{r+1} = \emptyset$ for each $r \in \{1,2,\ldots,2R-1\}$,

iii) $v(T_r) + v(T_{r+1}) = 1$ for each $r \in \{1,2,\ldots,2R-1\}$.

For a K-chain $(T_1,\ldots,T_{2R})$ we obviously have $v(T_1) = v(T_{2r+1})$ and $v(T_{2R}) = v(T_{2r})$ for each $r \in \{1,2,\ldots,R-1\}$. The equality $T_1 = T_{2R}$ now implies that the coalitions in a K-chain all have the same value in the game v. Because of iii) this value equals $\frac{1}{2}$. (3)

THEOREM 1. (Gurk(1959))

Let $v \in L_n$ be a three-valued game. If for each coalition S with $v(S) = \frac{1}{2}$ there exists a K-chain $(T_1,\ldots,T_{2R})$ with $S = T_1 = T_{2R}$ then v is extreme.

Proof. Suppose v is not extreme. Let w_1 and w_2 be elements of L_n, unequal to v, and scalar α, $0 < \alpha < 1$, such that $v = \alpha w_1 + (1-\alpha)w_2$. There is a coalition, say S', such that $v(S') \neq w_1(S')$. Then also $v(S') \neq w_2(S')$.
We already noticed that each game in K_n and, therefore, in L_n assigns nonnegative values to the coalitions. Thus, if for a coalition S we have $v(S) = 0 = \alpha w_1(S) + (1-\alpha)w_2(S)$ then w_1 and w_2 also assign 0 to S, i.e. $w_1(S) = w_2(S) = 0$. Therefore, $v(S') \neq 0$. Also $v(S') \neq 1$ since otherwise

$1 = \alpha w_1(S') + (1-\alpha)w_2(S')$, $w_1(S') \neq 1$ and $w_2(S') \neq 1$ which imply that one of the values $w_1(S')$ and $w_2(S')$ has to be greater than 1 which is impossible according to (2).

We conclude that $v(S') = \frac{1}{2}$ and $w_1(S') \neq \frac{1}{2}$ and $w_2(S') \neq \frac{1}{2}$. (4)

Therefore, there is a K-chain $(T_1,\ldots,T_{2R})$ with $S' = T_1 = T_{2R}$.

Let the index $r \in \{1,2,\ldots,2R-1\}$ be arbitrary.

The equality $v(T_r) + v(T_{r+1}) = 1$ implies that

$$\alpha(w_1(T_r) + w_1(T_{r+1})) + (1-\alpha)(w_2(T_r) + w_2(T_{r+1})) = 1. \tag{5}$$

Using the superadditivity property of w_j, $j = 1,2$, and $T_r \subseteq N \setminus T_{r+1}$ we gain

$$\begin{aligned} 1 = w_j(N) &= w_j(T_r) + w_j(N \setminus T_r) \\ &\geq w_j(T_r) + w_j(T_{r+1}), \quad j = 1,2. \end{aligned} \tag{6}$$

Combining (5) and (6) we conclude that

$$1 = w_j(T_r) + w_j(T_{r+1}), \quad j = 1,2.$$

Therefore, $(T_1,\ldots,T_{2R})$ is also a K-chain for the games w_1 and w_2. Thus, applying (3) we gain that $\frac{1}{2} = w_1(T_1) = w_1(S')$ which is in contradiction with (4). We conclude that the assumption that v is not extreme is false. □

The K-chain property, as formulated in theorem 1, is also sufficient when considering 5-person games :

THEOREM 2. (Gurk (1959))

A game v in L_5 is extreme if and only if for each coalition S with value v(S) unequal to 0 or 1 there exists a K-chain such that $S = T_1 = T_{2R}$. □

Theorem 2 obviously implies that the extreme elements of L_5 are three-valued games or simple games. There are extreme games in L_n, with $n > 5$, which assign more than three values to the coalitions. An example is given in Gurk (1959). Section 4 will also provide an example.

In Griesmer (1959) the reverse of theorem is shown :

THEOREM 3. (Griesmer (1959))

Let v be a three-valued game of L_n. If v is extreme in L_n then for each coalition S with value $\frac{1}{2}$ there exists a K-chain with $S = T_1 = T_{2R}$. □

The above theorems can also be used for the characterization of the three-valued extreme games of K_n. To show this each game v of K_n is identified with an (n+1)-person constant sum game v' of L_{n+1} defined by

$$v'(S) = \begin{cases} v(S) & \text{if } n+1 \notin S \\ 1-v(N\setminus S) & \text{if } n+1 \in S \end{cases} \quad \text{for each coalition } S \subseteq N \cup \{n+1\}. \quad (7)$$

In this context player n+1 is called the fictitious player.

THEOREM 4. (Spinetto (1971))

A game v of K_n is extreme if and only if v' is extreme in L_{n+1}. □

By using the results in Gurk (1959) on 5-person constant sum games, as displayed in theorem 2, and applying theorem 4 Spinetto was able to list the extreme superadditive games of K_4 (Spinetto (1971)). Of course, these games are three-valued or are simple games.

In Rosenmüller (1977) a completely different approach to the problem of characterizing (the) extreme games of K_n is proposed. The aim is to represent each game in K_n in such a way that the extremeness property can be translated into properties of the representation. This approach has been applied succesfully in the characterization of the extreme elements of the class of (0,1)-normalized convex games (Rosenmüller and Weidner (1974), Rosenmüller (1977)). Although the fact that convex games and superadditive games have similar definitions the representation approach for superadditive games has not been succesfull yet. For a detailed discussion of the used techniques and related subjects the reader is referred to Rosenmüller (1986).

4. UNIFORM GAMES

In this section we will introduce the notion of uniform games. In the definition of a uniform game essential coalitions play the central role. For a superadditive game v a coalition S is called *essential* whenever

$v(S) > v(T) + v(S \setminus T)$ for each non-empty coalition $T \subset S$.

This property for a coalition S may be seen as an incentive for the players in S to stay in cooperation because any split up of the coalition S will be unprofitable.

Let us denote with $B(v)$ the set of all essential coalitions of v. The superadditive game v is called *uniform* if all essential coalitions have the same value denoted by a_v, i.e. $v(S) = a_v$ for each $S \in B(v)$. For a uniform game v in K_n the value a_v is the smallest positive value that a coalition can admit in v.

Examples of uniform games are the superadditive simple games. For a superadditive simple game v we have $a_v = 1$ and $B(v)$ consists of all those coalitions S with

$v(S) = 1$ and $v(T) = 0$ for each coalition $T \subset S$.

These coalitions are also known as the minimal winning coalitions. Another example of a uniform game is the n-person superadditive game $\tilde{v}$ defined as (where $|S|$ denotes the number of players in the coalition S)

$$\tilde{v}(S) = \begin{cases} \frac{1}{2}|S| & \text{if } S \neq \emptyset \text{ and } |S| \text{ is even} \\ \frac{1}{2}(|S|-1) & \text{if } |S| \text{ is odd.} \end{cases} \qquad \text{for each } S \subset N.$$

Then $a_{\tilde{v}}$ equals 1 and the set $B(\tilde{v})$ consists of all two-person coalitions. Obviously the game $\tilde{v}_0 = (\tilde{v}(N)^{-1})\tilde{v}$ is also uniform and is an element of K_n. Note also that $\tilde{v}_0$ is a constant sum game whenever the number n is odd.

A superadditive game can easily be checked whether it is a uniform game. Furthermore, there is a simple characterization of the extreme superadditive games in K_n which are uniform and the remaining part of this section is devoted to this characterization.

LEMMA 5. Let v and w be two superadditive games.
If $v(S) \leq w(S)$ for all $S \in B(v)$ then $v \leq w$, i.e. $v(S) \leq w(S)$ for each $S \subseteq N$.

Proof. Suppose there is a coalition T with $v(T) > w(T)$. Let T be chosen such that it does not contain more players than any other coalition S with $v(S) > w(S)$. Of course, T cannot be an essential coalition for the game v. Hence, there is a non-empty coalition $S \subset T$ such that $v(T) = v(S) + v(T \setminus S)$.

According to the choice of T we have

$$w(S) + w(T \setminus S) \geqq v(S) + v(T \setminus S) = v(T) > w(T)$$

which is contradictory to the fact that w is superadditive. The lemma follows. □

In Shapley (1971) one may find an application of essential coalitions together with a decomposability criterion which will be used here also.
A game v is called *decomposable* if there is a coalition S such that

$$v(S) \neq 0,\ v(N \setminus S) \neq 0 \text{ and}$$

$$v(T) = v(T \cap S) + v(T \setminus S) \text{ for all } T \subseteq N. \tag{8}$$

A decomposable game v in K_n cannot be extreme for let coalition S be as in (8) and consider the games w_1 and w_2 defined as

$$w_1(T) = (v(S)^{-1})v(T \cap S) \text{ and } w_2(T) = (v(N \setminus S)^{-1})v(T \setminus S),\ T \subseteq N.$$

Then w_1 and w_2 are (0,1)-normalized superadditive games. Furthermore, w_1 and w_2 unequal v and

$v = v(S)w_1 + v(N \setminus S)w_2 = v(S)w_1 + (v(N)-v(S))w_2 = v(S)w_1 + (1-v(S))w_2$ which shows that v is not extreme.

A non-decomposable game of K_n may not be extreme either :
Consider the 3-person game v with

$$v(S) = 0 \text{ for each coalition } S \text{ with } |S| < 0,\ v(S) = \tfrac{1}{2} \text{ if } |S| = 2 \text{ and}$$

$$v(\{1,2,3\}) = 1.$$

One easily checks that v is an element of K_3 and non-decomposable (or apply lemma 6). Now let w_1 and w_2 be the simple games with

$w_1(S) = 1$ if $S \in \{\{1,2\},\{1,3\},N\}$ and $w_2(S) = 1$ if $S \in \{\{2,3\},N\}$. Then w_1 and w_2 are superadditive games, unequal to v, elements of K_3 and $v = \tfrac{1}{2}w_1 + \tfrac{1}{2}w_2$. Hence, v is not extreme.

A set B of coalitions is said to be *decomposable* if there exists a non-empty coalition $S \subset N$ such that

$$B \cap 2^S \neq \emptyset,\ B \cap 2^{N\setminus S} \neq \emptyset \text{ and}$$

$$B = (B \cap 2^S) \cup (B \cap 2^{N\setminus S}). \tag{9}$$

LEMMA 6. Let v be a superadditive game. Then v is decomposable if and only if B(v) is decomposable.

Proof. The "only if" part of the lemma is obviously true. Thus suppose that $B(v)$ is decomposable and let S be as in (9). Furthermore, let w_1 and w_2 be the game defined as

$w_1(T) = v(T \cap S)$ and $w_2(T) = v(T \setminus S)$, $T \subseteq N$.

Then, for each coalition $T \subseteq N$, $v(T) \geqq v(T \cap S) + v(T \setminus S) = w_1(T) + w_2(T)$.
Therefore, $v \geqq w_1 + w_2$.
On the other hand, for each essential coalition $T \in B(v)$ we have $T \subseteq S$ or $T \subseteq N \setminus S$ according to (9). This implies that $v(T) = w_1(T) + w_2(T)$.
Applying lemma 5 on the superadditive games v and $w_1 + w_2$ we gain that $v \leqq w_1 + w_2$. Hence, $v = w_1 + w_2$ and the lemma follows. □

THEOREM 7. Let v be a uniform game of K_n. Then v is extreme if and only if v is not decomposable.

Proof. The "only if" part follows immediately. Thus, let v be a game of K_n, uniform and not decomposable. Let $w_1, w_2 \in K_n$ and scalar α, $0 < \alpha < 1$, be such that $v = \alpha w_1 + (1-\alpha) w_2$. The theorem is proved whenever we have shown that $v = w_1 = w_2$. From lemma 6 it follows that $B(v)$ is not decomposable. Of course, $B(v)$ is non-empty and fullfills

if $S \subseteq T$ for two coalitions $S, T \in B(v)$ then $S = T$.

Therefore, $B(v)$ fullfills the conditions of the theorem mentioned in the appendix. Applying this theorem we state that there exists a sequence of essential coalitions, say $Q = (T_1, \ldots, T_R)$, such that

i) each essential coalition occurs at least once in Q; (11)

ii) the union of each two adjacent elements of Q does not contain two disjoint essential coalitions, i.e.
if $S, T \subseteq T_r \cup T_{r+1}$ for an r, $1 \leqq r \leqq R-1$ and $S, T \in B(v)$ then $S \cap T \neq \emptyset$. (12)

Of course, we have $v(T_r \cup T_{r+1}) \geqq v(T_r)$, $1 \leqq r \leqq R-1$. Suppose $v(T_r \cup T_{r+1}) > v(T_r) = a_v$ for an index $r \in \{1, 2, \ldots, R-1\}$. Consider the game v' which equals v on all coalitions except $T_r \cup T_{r+1}$ and $v'(T_r \cup T_{r+1}) = v(T_r)$. Since v is uniform and $v(T_r \cup T_{r+1}) > a_v$ the coalition $T_r \cup T_{r+1}$ cannot be essential. Therefore, $v'(S) = v(S)$ for each coalition S of $B(v)$. Furthermore, one easily checks with the help of (12) that

v' is superadditive. Applying lemma 5 we gain that $v' \geqq v$. Then $v(T_r) = v'(T_r \cup T_{r+1}) \geqq v(T_r \cup T_{r+1}) > v(T_r)$. A contradiction.
We conclude that $v(T_r \cup T_{r+1}) = v(T_r) = a_v$, $1 \leqq r \leqq R-1$.

Using $w_j(T_r \cup T_{r+1}) \geqq w_j(T_r)$, $j = 1,2$, $1 \leqq r \leqq R-1$, we gain

$$v(T_r) = \alpha w_1(T_r) + (1-\alpha) w_2(T_r)$$
$$\leqq \alpha w_1(T_r \cup T_{r+1}) + (1-\alpha) w_2(T_r \cup T_{r+1}) = v(T_r \cup T_{r+1}) = v(T_r).$$

This implies that $w_j(T_r \cup T_{r+1}) = w_j(T_r)$, $j = 1,2$, $1 \leqq r \leqq R-1$.
Analogously one proves that $w_j(T_r \cup T_{r+1}) = w_j(T_{r+1})$, $j = 1,2$, $1 \leqq r \leqq R-1$.
We conclude that w_1 and w_2 assign to each coalition in the sequence Q the same value, say a_1 and a_2. Of course, $\alpha a_1 + (1-\alpha) a_2 = a_v$.
Then $w_1(S) = a_1$ and $w_2(S) = a_2$ for each $S \in B(v)$ according to (11).
Therefore, $w_1(S) = (a_1/a_v)\, v(S)$ for each coalition S of B(v). Using lemma 5 we gain that $w_1 \geqq (a_1/a_v)\, v$. Similarly, $w_2 \geqq (a_2/a_v)\, v$. Then

$$v = \alpha w_1 + (1-\alpha) w_2 \leqq \alpha (a_1/a_v)\, v + (1-\alpha)(a_2/a_v)\, v$$
$$= a_v^{-1} (\alpha a_1 + (1-\alpha) a_2) v = v.$$

From this we conclude that $w_1 = (a_1/a_v)\, v$ and $w_2 = (a_2/a_v)\, v$.
Applying $w_1(N) = w_2(N) = v(N) = 1$ we gain that $a_1 = a_2 = a_v$. Hence, $w_1 = w_2 = v$ and, therefore, v is an extreme game. □

Now let us return to the examples of uniform games which are displayed at the beginning of this section : the simple games and the n-person game $\tilde{v}_0$.
The essential coalitions of a superadditive simple game have value 1.
This implies that each two different essential coalitions are not disjoint for otherwise there exists a coalition with value at least 2 which is impossible in a simple game. Therefore, the set B(v) of essential coalitions of a superadditive simple game v is non-decomposable. We conclude that the simple games in K_n are extreme.
In the game $\tilde{v}_0$ each two-person coalition is essential. Therefore, the set $B(\tilde{v}_0)$ is non-decomposable and, thus, $\tilde{v}_0$ is extreme in K_n. Of course, the game $\tilde{v}_0$ is, for $n > 5$, not simple and not three-valued. We mentioned already that, for n odd, $\tilde{v}_0$ is a constant sum game : $\tilde{v}_0 \in L_n$. Since $L_n \subset K_n$ and $\tilde{v}_0$ is extreme in K_n the game $\tilde{v}_0$ has to be extreme in L_n also, for n odd.

Theorem 7 does not characterize all extreme games of K_n. For example, in Spinetto (1971) the following 4-person extreme superadditive game may be found :

$v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{3,4\}) = v(\{1,2,3\}) = v(\{2,3,4\}) = \frac{1}{2}$, $v(\{1,2,4\}) = v(\{1,3,4\}) = v(\{1,2,3,4\}) = 1$ and $v(S) = 0$ for the other coalitions S.

This game is not uniform since $\{1,2\}$ and $\{1,2,4\}$ are essential coalitions of v with different values.

APPENDIX

ON A PROPERTY FOR A NON-DECOMPOSABLE FAMILY OF SUBSETS OF A FINITE SET.

Let N be a non-empty finite set. The family of all subsets of N is denoted by 2^N. Let $B \subseteq 2^N$ be a non-empty set. B is called *decomposable* if there is a subset S of N such that

$B \cap 2^S \neq \emptyset$, $B \cap 2^{N\setminus S}$ and

$B = (B \cap 2^S) \cup (B \cap 2^{N\setminus S})$.

THEOREM. Let $B \subseteq 2^N$, $B \neq \emptyset$. Furthermore, suppose that if

$S \subseteq T$ for two elements $S,T \in B$ then $S = T$. (1)

Then there exists a sequence $Q = (T_1,T_2,\ldots,T_R)$ of elements of B such that

i) each element of B occurs in Q, i.e.

for each $S \in B$ there is an r, $1 \leq r \leq R$, with $S = T_r$; (2)

ii) the union of each two adjacent elements of Q does not contain two disjoint elements of B, i.e.

if $S,T \subseteq T_r \cup T_{r+1}$ for an r, $1 \leq r \leq R-1$, and $S,T \in B$, then $S \cap T \neq \emptyset$. (3)

Proof. In the following we shall construct a sequence which has the desired properties. Note that for sequences which fullfill (3) each intersection of two adjacent elements is non-empty. Let $Q = (T_1,\ldots,T_R)$ be an arbitrary sequence of elements of B for which

$T_r \cap T_{r+1} \neq \emptyset$ for each r, $1 \leq r \leq R-1$, (4)

and suppose that there is an element of B which does not occur in Q. Let T be the union of the sets $T_1,\ldots,T_R$.

Consider the following two cases :

a) $B \cap 2^{N\setminus T} \neq \emptyset$.

Then there is an element, say S', of $B\setminus\{T_r : 1 \leqq r \leqq R\}$ such that $S' \cap T \neq \emptyset$.

b) $B \cap 2^{N\setminus T} \neq \emptyset$.

$(B \cap 2^T) \cup (B \cap 2^{N\setminus T}) \neq B$ since B is not decomposable by assumption. Therefore, there is an element, say S', of B such that $S' \cap T \neq \emptyset$ and $S' \cap N\setminus T \neq \emptyset$.

In both cases S' is chosen such that it does not occur in Q. Moreover, from $S' \cap T \neq \emptyset$ we achieve that there is an element T_r such that $S' \cap T_r \neq \emptyset$. Now the sequence $Q' = (T_1,\ldots,T_r,S',T_r,\ldots,T_R)$ fullfills (4) also and it contains one extra element of B.

We conclude that one may construct a sequence of elements of B in which each element of B occurs at least once and which fullfills (4) by applying the above method as many times as needed.

Now let $Q = (T_1,\ldots,T_R)$ be a sequence for which (2) and (4) holds and consider the index set

$$I(Q) = \{r : \text{there exist } S,T \in B \text{ with } S,T \subseteq T_r \cup T_{r+1} \text{ and } S \cap T = \emptyset\}.$$

Furthermore, let $d(Q) = \max\{\,|T_r \cup T_{r+1}| : r \in I(Q)\}$ and

$I_M(Q) = \{l \in I(Q) : |T_l \cup T_{l+1}| = d(Q)\}$.

If $I(Q) = \emptyset$ or, equivalently, $I_M(Q) = \emptyset$, then Q is a desired sequence.

Thus, suppose $I_M(Q) \neq \emptyset$ and let $S_l,S'_l \in B$, $S_l,S'_l \subset T_l \cup T_{l+1}$ and $S_l \cap S'_l = \emptyset$ for each $l \in I_M(Q)$.

Then, for each index l of $I_M(Q)$, we have :

If $S_l \cap T_l = \emptyset$ then $S_l \subset T_{l+1}$ which implies $S_l = T_{l+1}$ according to (1). Then, using $S_l \cap S'_l = \emptyset$, we conclude that $S'_l \subset T_l$ which implies $S'_l = T_l$. Thus, $\emptyset = S_l \cap S'_l = T_l \cap T_{l+1} \neq \emptyset$. A contradiction.

Therefore, $S_l \cap T_l \neq \emptyset$ and, similarly, $S_l \cap T_{l+1} \neq \emptyset$. (5)

Consider the sequence $Q' = (T'_1,\ldots,T'_L)$ which we obtain by inserting the element S_l between T_l and T_{l+1} in the sequence Q, for each $l \in I_M(Q)$.

Q' obviously fullfills (2) and, according to (5), also (4).

We will prove now that $d(Q') < d(Q)$.

For each $l \in I_M(Q)$ we have

$|T_l \cup S_l| \leqq |T_l \cup T_{l+1}| = d(Q)$ and $|S_l \cup T_{l+1}| \leqq d(Q)$ since

$S_l \subseteq T_l \cup T_{l+1}$.

If $|T_l \cup S_l| = d(Q) = |T_l \cup T_{l+1}|$ then we must have $T_{l+1} \backslash T_l \subseteq S_l$.

$S_l \cap S_l' = \emptyset$ now implies $S_l' \subseteq T_l$ and, therefore, $S_l' = T_l$ according to (1).

Thus, $S_l \subseteq T_{l+1}$, implying $S_l = T_{l+1}$. A contradiction.

Hence, $|T_l \cup S_l| < d(Q)$ and $|S_l \cup T_{l+1}| < d(Q)$ by the same reasoning. (6)

Using (6) and the construction of Q' from Q we conclude that $d(Q') < d(Q)$.

We described a method that constructs from a sequence Q, fullfilling the conditions (2) and (4) and $d(Q) > 0$, a sequence Q' for which also (2) and (4) holds and $d(Q') < d(Q)$. Now $d(Q)$ is a non-negative integer for each sequence Q with $I(Q) \neq \emptyset$.

Therefore, by applying the method finitely many times we obtain a sequence Q, fullfilling (2) and (4) and $I(Q) = \emptyset$.

Then Q is a sequence for which (2) and (3) holds.

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CHAPTER X

COMBINATORIAL GAMES

by Imma Curiel

1. INTRODUCTION

Combinatorial optimization problems are well-known in the literature. Examples of such problems are the traveling salesman problem, the optimal assignment problem and the minimum cost spanning tree problem. These problems can be viewed as describing situations where one person wants to maximize his revenues or minimize his costs and has to solve a combinatorial problem to do this. In the following we will consider combinatorial games. These games describe situations where there is a group of people each of which wants to maximize his profits or minimize his costs. Each has to solve a combinatorial problem to do this. These people can decide to combine their forces in order to have more profit or less costs. They can work together in the whole group or in subgroups. Each subgroup has to solve a combinatorial optimization problem in order to maximize its profits or minimize its costs. If they decide to work together in a group they have to decide how the profits or costs should be divided among the members of the group. Cooperative game theory proposes several methods for the division of profits or costs. In the following sections we will consider seven combinatorial games which are studied in literature. In the second section we give the necessary definitions of cooperative game theory. In the third section we introduce the seven games and give the most important results for them from the literature. In the fourth section we give a general method to generate core elements for five of these games and indicate why this method does not work for the two other games. In the last section we make some concluding remarks.

2. COOPERATIVE GAME THEORY

Cooperative game theory is used to describe and analyse situations in which a group of people can decrease their costs or increase their revenues by working together. Formally, a *cooperative game in characteristic function form* is an ordered pair $\langle N,v \rangle$ where N is a finite set, the *set of players* and v is the *characteristic function* which assigns to every subset S of N a real number v(S) with $v(\emptyset) = 0$. The set of subsets of N is denoted by 2^N, we call a subset of N a *coalition*. In this context v(S) is regarded as the worth of the coalition S, i.e. the revenue that the members of S can achieve if they work together. A cooperative game $\langle N,v \rangle$ is said to be *superadditive* if

$$v(S) + v(T) \leqq v(S \cup T) \text{ for all } S,T \in 2^N \text{ with } S \cap T = \emptyset \qquad (*)$$

and *convex* if

$$v(S) + v(T) \leqq v(S \cup T) + v(S \cap T) \text{ for all } S,T \in 2^N. \qquad (**)$$

The question which arises once the grand coalition N is formed is how v(N) should be divided among the players. Let $N = \{1,2,\ldots,n\}$, then an *allocation* of v(N) among the players can be described by a vector $x \in \mathbb{R}^n$ with $\sum_{i \in N} x_i = v(N)$. Here x_i denotes the amount allocated to player i. In the following we will denote $\sum_{i \in S} x_i$ for a vector $x \in \mathbb{R}^n$ and an $S \in 2^N \setminus \{\emptyset\}$ by x(S). *Solution concepts* assign an allocation or a set of allocations to a game.

The *core* is a solution concept which assigns a, possibly empty, set of allocations to a game. The core of $\langle N,v \rangle$ is denoted by C(v) and defined by

$$C(v) := \{x \in \mathbb{R}^n : x(N) = v(N),\ x(S) \geqq v(S) \text{ for all } S \in 2^N \setminus \{\emptyset\}\}.$$

If v(N) is allocated according to an element of C(v) no coalition has an incentive to split off the grand coalition because it cannot do better on its own. A convex game always has a non-empty core, cf. Shapley (23).

Let us imagine that the formation of N takes place according to a permutation π of N, with the players joining one after the other in the order $\pi^{-1}(1), \pi^{-1}(2),\ldots,\pi^{-1}(n)$. Let $P(\pi,i)$ denote the set of predecessors of i with respect to π, $P(\pi,i) := \{j \in N \mid \pi(j) < \pi(i)\}$. The marginal contribution of i with respect to π is denoted by $\psi_i^\pi(v)$ and defined by

$$\psi_i^{\pi}(v) := v(P(\pi,i) \cup \{i\}) - v(P(\pi,i)).$$

The *Shapely-value* Φ assigns the following allocation to $\langle N,v \rangle$:

$$\Phi_i(v) := \frac{1}{n!} \sum_{\pi \in \Pi_N} \psi_i^{\pi}(v) \qquad \text{for all } i \in N.$$

The Shapley-value gives player i his expected marginal contribution if all possible orders of formation are considered equally likely to occur. For each $S \in 2^N \setminus \{\emptyset\}$ we denote by Π_S the set of permutations of S. Even for a game with a non-empty core the Shapley-value may not be in the core. For a convex game the Shapley-value is the barycenter of the core.

For an allocation x we define the excess of a coalition S relative to x by

$$e(x,S) := v(S) - x(S).$$

Schmeidler (21) has defined the *nucleolus* of v as that allocation which minimizes the maximum excess lexicographically. Schmeidler has proved that there is a unique allocation for which the lexicographical minimum is achieved. If the core of a game is non-empty then it contains the nucleolus.

The last solution concept that we treat here is the *τ-value* introduced by Tijs (28). For a cooperative game $\langle N,v \rangle$ we define

$$M_i^v = v(N) - v(N \setminus \{i\}) \text{ for all } i \in N.$$

M_i^v is the maximum payoff that player i can expect to obtain, if he asks for more, the others will do better by working without him. Let S be a coalition with $i \in S$, suppose all members of $S \setminus \{i\}$ get their maximum then what is left for player i is equal to

$$R^v(S,i) := v(S) - M^v(S \setminus \{i\}).$$

The minimum that i will consent to get is

$$\Gamma_i^v := \max_{S \ni i} R^v(S,i),$$

because he can ensure himself this by offering the members of a coalition S, for which the maximum is achieved, their maximum payoff and remaining with Γ_i^v. A cooperative game $\langle N,v \rangle$ is said to be *quasi-balanced* if $\Gamma^v(N) \leq v(N) \leq M^v(N)$. For a quasi-balanced game the τ-value is defined by

$$\tau_i^v := \lambda M_i^v + (1-\lambda)\Gamma_i^v \quad \text{for all } i \in N.$$

Here λ is uniquely determined by $\tau(N) = v(N)$.

The notions that we have introduced have all been stated in terms of revenues but in an analogous way we can state them in terms of costs. The characteristic function of a cost game will be denoted by c instead of v. For a cost game the notions of subadditivity and concavity are important. A game is subadditive (concave) if it satisfies (*) ((**)) with the inequalities reversed.
For the definition of the core of a cost game we also have to reverse the inequalities. The Shapley-value of a cost game is defined in the same way as the Shapley-value of a revenue game. The nucleolus of a cost game is defined to be that allocation which maximizes the minimum excess lexographically. For a cost game $\langle N,c \rangle$ we define M_i^c in the same way as for a revenue game. But here M_i^c is considered to be the minimum cost that i can expect that he will have to pay. $R^c(S,i)$ can also be defined in the same way. For the definition of Γ_i^c we have to change "max" into "min", Γ_i^c is considered to be the maximum that player i will consent to pay. A cost game $\langle N,c \rangle$ is said to be quasi-balanced if $\Gamma^c(N) \geqq c(N) \geqq M^c(N)$. The τ-value of $\langle N,c \rangle$ is defined with the aid of M_i^c and Γ_i^c in the same way as the τ-value of a revenue game.

In the following we will denote the worth of a coalition $\{i,j,\ldots,k\}$ by $v(i,j,\ldots,k)$ instead of $v(\{i,j,\ldots,k\})$. The same holds for the cost of the coalition.

With every permutation $\pi \in \Pi_N$ a *permutation matrix* $P = [p_{ij}]_{i=1,j=1}^{n\ \ n}$ is associated with $p_{ij} \in \{0,1\}$ for all $i,j \in N$ and $p_{ij} = 1$ if and only if $\pi(i) = j$. Such a permutation matrix $[p_{ij}]_{i=1,j=1}^{n\ \ n}$ satisfies the following conditions :

$$\sum_{j=1}^{n} p_{ij} = 1 \qquad \text{for all } i \in N,$$

$$\sum_{i=1}^{n} p_{ij} = 1 \qquad \text{for all } j \in N,$$

$$p_{ij} \in \{0,1\} \qquad \text{for all } i,j \in N.$$

If we change the last condition into $p_{ij} \geqq 0$ for all $i,j \in N$ we get a larger set of matrices. These matrices are called *doubly stochastic*.

3. COMBINATORIAL GAMES

A cooperative game is said to be a combinatorial game if for every $S \in 2^N \setminus \{\emptyset\}$ the worth or cost of S is given by the value of a combinatorial optimization problem which depends on S. In this section we will study seven different combinatorial games which appear in literature and describe the results obtained for these games.

Minimum cost spanning tree games.

Claus and Kleitman (2) studied the following situation. Several customers who are geographically separated have to be linked to a certain supplier. We can think, for example, of the supplier being an electricity plant and the customers being firms or cities which want to make use of the service provided by the supplier. A user can be linked directly to the supplier or via other users. Let us denote the set of users by $N = \{1,2,\ldots,n\}$ and the supplier by 0. A link between elements i and j of $N \cup \{0\}$ we denote by l_{ij}. Every link induces a non-negative cost. The cost of link l_{ij} we denote by $k(l_{ij})$. Note that $l_{ji} = l_{ij}$. If the customers cooperate they can try to find a cheapest way to connect them all with the supplier. Let us think of the customers and the supplier as vertices of a complete graph G_N. We denote the edge set of this graph by E_N. Every edge corresponds to a link l_{ij} with $i,j \in N \cup \{0\}$ and vice versa. To edge l_{ij} we attach cost $k(l_{ij})$. Then the problem of finding a cheapest way to connect all customers with the supplier is equivalent to the problem of finding a minimum cost spanning tree of the graph G_N. The question that Claus and Kleitman raised is how to allocate this cost among the customers. They studied several methods to allocate this cost and discussed their pro's and contra's. Further, they gave a list of desirable criteria for a cost allocation.

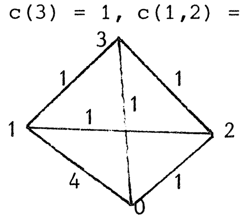
Bird (1) applied cooperative game theory to this situation. A cooperative game $\langle N,c \rangle$ can be defined as follows. The set of players is equal to the set of customers. For every coalition S we define c(S) to be the cost of linking all members of S to the supplier in a cheapest possible way without making links which involve non-members of S. Hence c(S) is equal to the cost of a minimum cost spanning tree in the complete graph G_S with set of vertices equal to $S \cup \{0\}$. We denote the edge set of this graph by E_S. The game $\langle N,c \rangle$ is called a *minimum cost spanning tree game*, cf. Granot and Huberman (11). Bird proved that this game has a non-empty core.

THEOREM (Bird). Let $\langle N,c \rangle$ be a minimum cost spanning tree game. Then $C(c) \neq \emptyset$.

PROOF. Let T_N be a minimum cost spanning tree of G_N. For every $i \in N$ let e_i be the edge on the unique path from 0 to i in T_N incident to i. We construct an element x of the core in the following way. Let $x_i = k(e_i)$ for every $i \in N$. Then $x(N) = \sum_{i \in N} k(e_i) = c(N)$. For $S \in 2^N \setminus \{\emptyset\}$ let T_S be a minimum cost spanning tree of G_S. For every $i \in S$ let e_i^S be the edge on the unique path from 0 to i in T_S incident to i. Then $c(S) = \sum_{i \in S} k(e_i^S)$. Suppose $x(S) > c(S)$, then there is an $i \in S$ such that $k(e_i^S) < k(e_i)$. Remove e_i from T_N and add e_i^S, this results in a spanning tree of G_N with lesser cost than T_N which contradicts the fact that T_N is a minimum cost spanning tree of G_N. Hence $x(S) \leqq c(S)$ for every $S \in 2^N \setminus \{\emptyset\}$ and it follows that $x \in C(c)$. □

The following example illustrates the fact that a minimum cost spanning tree game needs not be concave.

EXAMPLE. Let $N = \{1,2,3\}$. The costs are given in the following figure. Then $c(1) = 4$, $c(2) = 1$, $c(3) = 1$, $c(1,2) = c(1,3) = c(2,3) = 2$, $c(1,2,3) = 3$.



We see that $c(1,2) + c(1,3) = 4 < 7 = c(1) + c(1,2,3)$ so the game is not concave. This game has core $C(c) = \{(1,1,1)\}$. The Shapley-value $\varphi(c) = (2,\frac{1}{2},\frac{1}{2}) \notin C(c)$. Bird considers a class of weighted Shapley-values which

lie in the core of minimum spanning tree games. He abandons the assumption that all permutations of N are equiprobable. Only feasible permutations may have non-zero probability. A permutation $\pi \in \Pi_N$ is called *feasible* if there exists a minimum cost spanning tree T_N of G_N such that G_N contains minimum cost spanning trees of G_S for all coalitions S of the form $S = \{\pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(q)\}$ with $q \in \{1, \ldots, n\}$. Note that every minimum cost spanning tree T_N of G_N induces at least one feasible permutation $\pi \in \Pi_N$. In the definition of π we only have to make sure that if j is on the unique path from 0 to i then $\pi(i) > \pi(j)$. Bird proves that if we assign probability $p_\pi = 0$ to π whenever π is not feasible and probability $p_\pi \geqq 0$ if π is feasible with the sum of p_π over all feasible π equal to 1, then the weighted Shapley-values defined with these probabilities are always contained in the core of a minimum cost spanning tree game.
Granot and Huberman (12) have introduced the notion of a *permutationally concave game*. A game $\langle N,c \rangle$ is called permutationally concave if there exists a permutation π such that for all $U \in 2^N$ and $S \subset T \subset N \setminus U$ with S and T of the form $\{\pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(q)\}$ where $q \in \{1, \ldots, n\}$ the following holds

$$c(S \cup U) - c(S) \geqq c(T \cup U) - c(T) \qquad (*).$$

It follows that every concave game is permutationally concave. Granot and Huberman prove that the core of a permutationally concave game is not empty and that minimum cost spanning tree games are permutationally concave. The results of Bird and Granot and Huberman are related in the sense that if $\langle N,c \rangle$ is a minimum cost spanning tree game then (*) holds for every feasible permutation. Further, for every permutationally concave game c, if π is a permutation for which (*) holds then the marginal vector $m^\pi(c)$ is an element of C(c). Hence every convex combination of such $m^\pi(c)$'s is an element of C(c). This implies that for minimum cost spanning tree games every convex combination of the marginal vectors corresponding to feasible permutations is an element of the core and the result of Bird concerning the class of weighted Shapley-values follows.
Granot and Huberman also prove : if a minimum cost spanning tree game can be decomposed into q games $\langle N_k, c_k \rangle$, where the N_k's form a partition of N, then the core and the nucleolus of $\langle N,c \rangle$ is the Cartesian product of the cores and nucleoli of the games $\langle N_k, c_k \rangle$.

Megiddo (15) has proved that if the customers may use other arcs besides the ones connecting two customers or a customer and the supplier the resulting game can have an empty core.

In (16) Megiddo studies a class of games related to minimum cost spanning tree games. In these games a spanning tree T_N of G_N is given and the cost $c(S)$ of a coalition S is the total cost of edges that belong to some path from 0 to a vertex $i \in S$. This game is, in fact, not a combinatorial game as the computation of the characteristic function c does not involve a combinatorial optimization problem. Megiddo shows that the core of this game is not empty and gives algorithms to compute the nucleolus within $O(n^3)$ operations and the Shapley-value within $O(n)$ operations. One can prove in a straightforward way that this game is concave.

Discrete flow games

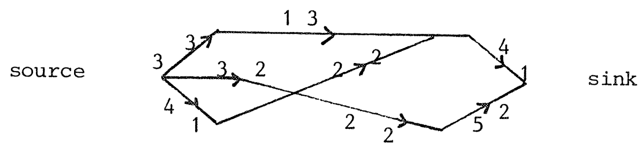
We consider the following situation. A directed network G is given through which discrete quantities of a product have to be transported. Let P be the set of vertices of G and L the set of arcs. Every arc $l \in L$ has a certain capacity $c(l) \in \mathbb{N}$ which denotes the maximum quantity that can be transported through this arc. Let $N = \{1,2,\ldots,n\}$ be the set of players. Every arc belongs to a player. Two vertices are distinguished from the others. One is called the *source*, the other the *sink*. A *discrete flow* from source to sink in this network is a function f from L to $\mathbb{N}$ with $f(l) \leqq c(l)$ for every $l \in L$ and such that for every vertex p except the source and the sink $\Sigma\{f(l) : l \text{ starts at } p\} = \Sigma\{f(l) : l \text{ ends at } p\}$. The *value* of such a flow is $\Sigma\{f(l) : l \text{ ends at the sink}\} = \Sigma\{f(l) : l \text{ starts at the source}\}$. A discrete flow with maximal value is called a *maximum discrete flow*. Let A be a subset of P such that the source is an element of A and the sink is not an element of A. By $(A, P \setminus A)$ we denote the subset of L consisting of arcs which have as their starting point an element of A and as their endpoint an element of $P \setminus A$. Such a subset of L is called a *cut* of the network G. The capacity of a cut is the sum of the capacities of its members. A well-known result of Ford and Fulkerson states that the value of a maximum discrete flow is equal to the capacity of a minimum cut, i.e. a cut with minimum capacity. A cooperative game

$\langle N,v \rangle$ can be defined as follows. For every $S \in 2^N \setminus \{\emptyset\}$ the worth $v(S)$ of S is the value of a maximum discrete flow f from source to sink with $f(l) = 0$ if l does not belong to a member of S. Such a game is called a *discrete flow game*. These games are discretizations of the flow games introduced by Kalai and Zemel in (13). Kalai and Zemel proved that flow games have non-empty cores. The following theorem states that discrete flow games have non-empty cores. The proof runs in the same way as the proof of Kalai and Zemel.

THEOREM. Let $\langle N,v \rangle$ be a discrete flow game. Then $C(v) \neq \emptyset$.

PROOF. Let $(A, P \setminus A)$ be a minimum cut in the network G of the game $\langle N,v \rangle$. We define an $x \in \mathbb{R}^n$ as follows. For every $i \in N$ x_i is the sum of the capacities of the arcs in $(A, P \setminus A)$ which belong to i, if i does not own any arc in $(A, P \setminus A)$ then $x_i = 0$. We prove that $x \in C(v)$. Because $v(N)$ is equal to the capacity of $(A, P \setminus A)$ it follows that $x(N) = v(N)$. Let $S \in 2^N \setminus \{\emptyset\}$, then $v(S)$ is the value of the maximum discrete flow in the network G_S which we construct from G by deleting all arcs which do not belong to a member of S. Then the restriction of $(A, P \setminus A)$ to the network G_S is a cut in G_S. It follows that $v(S)$ is less than the capacity of the restriction of $(A, P \setminus A)$ to G_S which equals $x(S)$ and we have proved that $x \in C(v)$. □

EXAMPLE. Let $N = \{1,2,3\}$. The network G is given in the following figure. The first number gives the capacity of an arc, the second the owner.



Here $v(1) = v(3) = 0$, $v(2) = 2$, $v(1,2) = 4$, $v(1,3) = 1$, $v(2,3) = 2$ and $v(1,2,3) = 5$. The core element corresponding to the minimum cut is $(0,4,1)$. $C(v)$ is the convex hull of the points $(2,2,1)$, $(3,2,0)$, $(0,4,1)$ and $(1,4,0)$.

Assignment and permutation games.

Shapley and Shubik (25) have considered the following market. Let there be m merchants who want to sell one indivisible commodity each, e.g. a house and n buyers who want to purchase one of the indivisible commodities each. Let M be the set of merchants and B the set of buyers. A merchant $j \in M$ values his house at c_j units of money. For all $i \in B$ and $j \in M$ we denote the value for buyer i of the house of merchant j by h_{ij}. A buyer $i \in B$ and a merchant $j \in M$ can try to cooperate and make some profit. If j values his house more than i does then they can make no profit. If i values the house of j more than j does then they can make a joint profit of $h_{ij} - c_j > 0$. For all $i \in B$ and $j \in M$ we define a_{ij} to be the profit they can make i.e.

$$a_{ij} := \max\{0, h_{ij} - c_j\}.$$

The *assignment game* $\langle B \cup M, v \rangle$ corresponding to this market is defined as follows. For every $S \in 2^N \setminus \{\emptyset\}$

$$v(S) := \max(a_{i_1 j_1} + a_{i_2 j_2} + \dots + a_{i_r j_r})$$

where $r = \min\{|S \cap B|, |S \cap M|\}$ and the maximum is taken over all assignments of players $j_1, \dots, j_r$ in $S \cap M$ to players $i_1, \dots, i_r$ in $S \cap B$. Here and in the rest of this paper we take the maximum (minimum) over the empty set to be equal to zero. This implies that $v(S) = 0$ for $S \subset M$ or $S \subset B$.

A game which is closely related to the assignment game is the *permutation game* introduced by Tijs et al. (29). They consider a situation in which n persons all have one job to be processed and one machine on which each job can be processed. No machine is allowed to process more than one job. If player i processes his job on the machine of player j then the processing cost equals c_{ij}. Let $N = \{1,2,\dots,n\}$ be the set of players. The permutation game $\langle N,c \rangle$ is defined by

$$c(S) := \min_{\pi_S \in \Pi_S} \sum_{i \in S} c_{i\,\pi(i)} \quad \text{for all } S \in 2^N.$$

Although the assignment game is defined in terms of revenues and the situation it describes is typically bipartite while the permutation game is defined in terms of costs and generally there is no bipartiteness in the situation it describes, the two games are closely related. In Tijs et al. (29) and in Curiel and Tijs (5) several relations between the two games are studied. Both games have a non-empty core. This can be proved in several ways for both games. All of these proofs use the following theorem of Birkhoff-von Neumann on the extreme points of doubly stochastic matrices.

THEOREM (Birkhoff-von Neumann). The extreme points of the set of $n \times n$-doubly stochastic matrices are exactly the $n \times n$-permutation matrices. □

As an illustration we give one of the proofs of the non-emptiness of the core of a permutation game. This proof is similar to the proof of Shapley and Shubik of the non-emptiness of the core of the assignment game and can be found in Curiel and Tijs (5).

THEOREM (Tijs et al. (29)). Permutation games have a non-empty core.
PROOF. Let $\langle N,c \rangle$ be a permutation game with costs c_{ij} for $i,j \in N$. We consider the problem of determining $c(N)$. That is the following integer programming problem

$$\min \sum_{i,j \in N} c_{ij} x_{ij} \text{ subject to}$$

$$\sum_{i \in N} x_{ij} = 1 \text{ for all } j \in N, \quad \sum_{j \in N} x_{ij} = 1 \text{ for all } i \in N,$$

$$x_{ij} \geq 0 \text{ and } x_{ij} \in \{0,1\} \text{ for all } i,j \in N.$$

Here $x_{ij} = 1$ indicates that player i uses the machine of player j. Because of the Birkhoff-von Neumann theorem this problem is equivalent to the linear programming problem with the integer conditions left out. So from the duality theorem of linear programming it follows that $c(N)$ is equal to the value of the dual problem which is

$$\max \sum_{i \in N} y_i + \sum_{i \in N} z_i \text{ subject to}$$

$$y_i + z_j \leq c_{ij} \text{ for all } i,j \in N.$$

Let $(y_1,\ldots,y_n,z_1,\ldots,z_n)$ be an optimal solution of this dual problem then $\sum_{i \in N} (y_i + z_i) = x(N)$ and $\sum_{i \in S} (y_i + z_i) \leq \sum_{i \in S} c_{i\pi_S(i)}$ for all permutations π_S of S, so $\sum_{i \in S} (y_i + z_i) \leq c(S)$ for all $S \in 2^N$ and it follows that

$(y_1 + z_1, y_2 + z_2, \ldots, y_n + z_n) \in C(c)$. □

The core element described in this theorem corresponds in the following way to a price mechanism. Let every player $i \in N$ ask a price $-z_j$ for his machine. To minimize his cost player i will look for the minimum of $c_{ij} - z_j$ over all $j \in N$. For all $j \in N$ we have $c_{ij} - z_j \geqq y_i$ and from the fact that $(y_1,\ldots,y_n,z_1,\ldots,z_n)$ is optimal for the dual problem it follows that for at least one $j \in N$ equality holds. So the cost of player i will be y_i minus the price he gets for his machine which equals $-z_i$. His total cost will be $y_i - (-z_i) = y_i + z_i$ which is exactly the cost allocated to him according to the core element.

EXAMPLE. Let $N = \{1,2,3\}$ and let $\langle N,c \rangle$ be the permutation game with the following costs : $c_{11} = 1$, $c_{12} = 4$, $c_{13} = 5$, $c_{21} = 2$, $c_{22} = 8$, $c_{23} = 10$, $c_{31} = 5$, $c_{32} = 6$, $c_{33} = 7$. Then $c(1) = 1$, $c(2) = 8$, $c(3) = 7$, $c(1,2)=6$, $c(1,3) = 8$, $c(2,3) = 15$ and $c(1,2,3) = 13$. The core $C(c)$ of this game is the convex hull of $(-2,8,7)$ and $(1,5,7)$.

Curiel and Tijs (5) studied extensions of assignment games and permutation games, tridimensional assignment games and bipermutation games, respectively. In these games, there occur two types of indivisible goods. Curiel and Tijs proved that these games can have an empty core. They gave two classes of these games for which the members have a non-empty core. Shapley and Scarf (24) study a game without side payments arising from a market with indivisibilities. They prove that this game has a non-empty core and that it is always possible to find competitive prices in the market. Wako (30) also studies a market with indivisible goods. Quinzii (19), Gale (8) and Wako (31) study a market with indivisible goods and a perfect divisible good which can be regarded as money.

Sequencing games

Sequencing games have been introduced by Curiel et al. (4). They consider the following situation. Let there be n customers waiting in a queue before a counter. Each customer $i \in N = \{1,2,\ldots,n\}$ has a certain *service time* $s_i > 0$ and a cost function $c_i : \mathbb{R}_+ \to \mathbb{R}$. For each customer i his cost, if his waiting time plus service time equal t, is given by $c_i(t)$. The original position of customer i in the queue is given by $\sigma(i)$ where σ is a permutation of N. Let $P(\sigma,i)$ denote the set of predecessors of i with respect to σ, $P(\sigma,i) := \{j \in N \mid \sigma(j) < \sigma(i)\}$. Then the total cost of a coalition $S \in 2^N \setminus \{\emptyset\}$ if everyone is served according to σ is given by

$$C_\sigma(S) = \sum_{i \in S} c_i \left(\sum_{j \in P(\sigma,i)} s_j + s_i \right).$$

By rearranging their positions before the counter the customers can decrease the total cost of N. The *sequencing game* $\langle N,v \rangle$ is defined as follows. The worth $v(S)$ of a coalition $S \in 2^N \setminus \{\emptyset\}$ is defined to be equal to the maximal cost savings that S can obtain by rearranging its members. But not all rearrangements are allowed, S may only rearrange its members in such a way that two members who have a non-member between them may not change position. Curiel et al. prove that if the cost functions are linear then the sequencing game as defined above is convex and hence it has a non-empty core. They introduce a division method which generates an element of the core. They also give expressions for the Shapley-value and the τ-value in this case.

If it is allowed for a coalition S to jump over non-members in the process of rearranging its members the resulting game can have an empty core.

EXAMPLE. Let $N = \{1,2,3\}$, $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 3$, $s_1 = 3$, $s_2 = 1$, $s_3 = 2$, $\alpha_1 = 6$, $\alpha_2 = 5$, $\alpha_3 = 6$. Then $C_\sigma(N) = 74$. An optimal order would be 2 first then 3 and 1 last. For the sequencing game $\langle N,v \rangle$ we have $v(i) = 0$, for all $i \in N$, $v(1,2) = 9$, $v(1,3) = 0$, $v(2,3) = 0$ and $v(1,2,3) = 15$. $C(v)$ is the convex hull of (9,0,6), (0,9,6), (15,0,0) and (0,15,0).

Traveling salesman games.

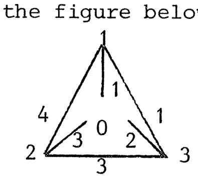
In (18) Potters et al. consider the following problem. A speaker is invited by several universities to deliver a talk. How should his travel costs from his home town along all the universities and back be allocated among the universities ? To analyse this situation they consider two types of games. Let us denote the home town of the speaker by 0 and the universities by 1,2,...n. Let $N = \{1,2,\ldots,n\}$. For all $i,j \in N \cup \{0\}$ we denote the travel cost from i to j by c_{ij}. It is assumed that $c_{ij} \geq 0$ for all $i,j \in N$ and that the triangle inequality holds, i.e. $c_{ij} + c_{jk} \geq c_{ik}$ for all $i,j,k \in N$. The *traveling salesman game* $\langle N,c \rangle$ is defined as follows. For every $S \in 2^N \setminus \{\emptyset\}$

$$c(S) := \min_{\pi_S} \sum_{i \in S \cup \{0\}} c_{i\,\pi_S(i)}$$

where the minimum is taken over all cyclic permutations of $S \cup \{0\}$. Potters et al. show that if the costs are not symmetric, i.e. $c_{ij} \neq c_{ji}$ for certain $i,j \in N$ then the traveling salesman game as defined above can have an empty core. In the case of symmetric costs it is still an open question whether or not the game always has a non-empty core.

In the traveling salesman game for every $S \in 2^N \setminus \{\emptyset\}$ the cost $c(S)$ is equal to the cost of a traveling salesman tour with minimal cost along the members of $S \cup \{0\}$. The other game that Potters et al. define is in fact not a combinatorial game because only the cost of N is defined by a combinatorial optimization problem. Let σ be a cyclic permutation of $N \cup \{0\}$ which defines a traveling salesman tour with minimal cost on $N \cup \{0\}$. Then the game $\langle N,c_\sigma \rangle$ is defined as follows. For every $S \in 2^N \setminus \{\emptyset\}$, $c_\sigma(S)$ is equal to the cost of the tour on $S \cup \{0\}$ which results when all $i \in N \setminus S$ are skipped and the members of S are visited in the same order as in the tour on $N \cup \{0\}$ defined by σ. Potters et al. have proved that this game has a non-empty core. For a certain class of cost matrices $[c_{ij}]_{i=0,j=0}^{n\ \ n}$ the traveling salesman game $\langle N,c \rangle$ coincides with a game $\langle N,c_\tau \rangle$ for a certain cyclic permutation τ and hence for this class the traveling salesman game has a non-empty core.

EXAMPLE. Let $N = \{1,2,3\}$. Let $\langle N,c \rangle$ be the symmetric traveling salesman game with costs as given in the figure below.



Then $c(1) = 2$, $c(2) = 6$, $c(3) = 4$, $c(1,2) = 8$, $c(1,3) = 4$, $c(2,3) = 8$ and c(1,2,3) = 8. An optimal tour is to visit 1 first then 3 then 2 and return to 0. C(c) is the convex hull of (0,6,2), (0,4,4), (2,6,0) and (2,2,4).

Location games

Tamir (26) considered the following situation. A graph G with set of vertices V and edge set E is given. Each edge has a certain positive length. The length of a path in G is the sum of the lengths of the edges that belong to the path. Let v_1 and v_2 be two vertices of G. The distance $d(v_1,v_2)$ between v_1 and v_2 is defined to be the length of a shortest path from v_1 to v_2. Two subsets N and Q of V are given. $N = \{1,...,n\}$ is the set of players. Each player is considered to be located in the corresponding vertex. Service centers have to be located in G in order to provide service to the players. The set $Q = \{q_1,...,q_t\}$ denotes the possible locations for service centers. The cost of establishing a center at q_j is $c_j \geqq 0$. Player $i \in N$ demands that at least one center will be located at a distance of at most $r_i \geqq 0$ from him. The problem is to locate the centers in such a way that all the demands are fulfilled and the cost is minimized. It is assumed that all the demands can be met. The *location game* $\langle N,c \rangle$ is defined as follows. Every $S \in 2^N \setminus \{\emptyset\}$ wants to fulfil the demands of its members, c(S) is the minimum cost needed to do this. Tamir shows that in general this game can have an empty core. If the graph G is a tree however, the game has a non-empty core.

THEOREM (Tamir). Let $\langle N,c \rangle$ be a location game defined on a tree G. Then $C(v) \neq \emptyset$.

PROOF. Define the $n \times t$ - matrix $A = [a_{ij}]_{i=1,j=1}^{n\ \ t}$ as follows.

$$a_{ij} = \begin{cases} 1 \text{ if } d(i,q_j) \leqq z \\ 0 \text{ otherwise.} \end{cases}$$

Then for every $S \in 2^N \setminus \{\emptyset\}$ the cost c(S) is given by the value of the problem

$$\min c \,.\, x \quad \text{subject to}$$
$$Ax \geqq e^S$$
$$x \in \{0,1\}.$$

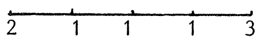
Here $e^S \in \mathbb{R}^n$ is the vector with e_i^S equal to 1 if $i \in S$ and e_i^S equal to 0 otherwise. This problem is equivalent to the linear programming problem with the 0-1 constraints replaced by $x \geqq 0$, cf. Tamir (27). From the duality theory of linear programming it follows that c(S) is equal to the value of the problem

$$\max y \cdot e^S \text{ subject to}$$
$$yA \leqq c$$
$$y \geqq 0.$$

Let $\hat{y}$ be an optimal solution for the problem with objective function e^N. Then $\hat{y}$ is feasible for all the problems. Hence for every $S \in 2^N \setminus \{\emptyset\}$ we have $c(S) \geqq \hat{y}(S)$ and it follows that $\hat{y} \in C(c)$. □

Tamir proves in fact that the core is equal to the set of optimal solutions of the dual problem with objective function e^N.

EXAMPLE. Let $N = Q = \{1,2,3\}$. G is the tree given below, the distances are given in the figure.



Let $c_1 = 3$, $c_2 = 1$, $c_3 = 2$, $r_1 = 1$, $r_2 = 1$, $r_3 = 1$. Then the location game $\langle N,c \rangle$ is given by $c(1) = 1$, $c(2) = 1$, $c(3) = 2$, $c(1,2) = 1$, $c(1,3) = 2$, $c(2,3) = 3$, $c(1,2,3) = 3$ and $C(c) = \{(0,1,2)\}$.

4. GENERAL APPROACH

If we consider the discrete flow game, the assignment game, the permutation game and the location game we see that the existence of the core has been proved for all of these games in a similar way. For all these games the cost or worth of a coalition S can be found by minimizing or maximizing a linear function which does not depend on S, on the integer points of a polytope P^S. In all these cases an integer optimal solution can be found without requiring integrality explicitly. So c(S) or v(S) can be computed by solving a linear optimization problem. By the duality theorem of linear programming it follows that this problem and its dual have the same value. For all S this dual problem has the same feasible region so an optimal solution $\hat{y}$ for the dual problem with N is feasible for all the other dual problems and a core element can be generated by $\hat{y}$. The following general model contains all the four games mentioned above. We will describe it for a cost game. Let $\langle N,c \rangle$ be a cost game such that for all $S \in 2^N \setminus \{\emptyset\}$

$$c(S) = \min_{\substack{x \in P^S \\ x \text{ integer}}} L(x).$$

Here L is a linear function which does not depend on S. Suppose that the minimum does not change if the integrality condition is left out. Then c(S) is equal to the value of the dual problem, say

$$c(S) = \max_{y \in R} H^S(y).$$

Here H^S is a linear function which depends on S and R is a polyhedron which does not depend on S. Further, $H^S(y) = \sum_{i \in S} H^i(y)$ for every $S \in 2^N \setminus \{\emptyset\}$, $y \in R$. Let $\hat{y} \in R$ be such that $c(N) = H^N(\hat{y})$. Define $z \in \mathbb{R}^n$ by $z_i = H^i(\hat{y})$ for every $i \in N$. Then $z(N) = H^N(\hat{y}) = c(N)$ and $z(S) = H^S(\hat{y}) \leqq c(S)$ for all $S \in 2^N \setminus \{\emptyset\}$ and we see that $z \in C(c)$. The moment the integrality conditions are dropped everything we do is similar to the procedure Owen (17) follows in proving that a linear production game has a non-empty core. We can generalize this model a little bit more by replacing the assumption that $H^S(y) = \sum_{i \in S} H^i(y)$ by the assumption that for each $y \in R$ the cost game H_y defined by $H_y(S) = H^S(y)$ for every $S \in 2^N \setminus \{\emptyset\}$, has a non-empty core, cf. Granot (10) and Curiel et al. (3). Let $\hat{y}$ be as above and let $w \in C(H_{\hat{y}})$. Then $w(S) \leqq H_{\hat{y}}(S) = H^S(\hat{y}) \leqq c(S)$ and it follows that $w \in C(c)$.

In the case of the location game we have seen that the whole core is generated by optimal solutions of the dual problem. This is also true for the assignment game. Samet and Zemel (20) discuss conditions which are fulfilled by these games and which guarantee that the core is equal to the set generated by optimal dual solutions.

If we try to treat the traveling salesman game in the way described above we encounter the difficulty that the integrality conditions which arise in the computation of c(S) cannot be left out because the polytope P^S that we obtain has non-integer extreme points. This is also the case

For the minimum cost spanning tree game c(S) is the value of the following integer programming problem

$$\min \sum_{i \in N \cup \{0\}} \sum_{j \in N \cup \{0\}} k(l_{ij})\, x_{ij} \quad \text{subject to}$$

$$\sum_{i \in S \cup \{0\}} \sum_{j \in S \cup \{0\}} x_{ij}\, 1_T(i)(1 - 1_{T(j)}) \geqq 1 \quad \text{for all } \emptyset \neq T \subset S$$

$$x_{ij} \geqq 0,\ x_{ij} \text{ integer} \quad \text{for all } i,j \in N \cup \{0\}$$

Here 1_T is the characteristic function of T, i.e. $1_T(i) = 1$ if $i \in T$, $1_T(i) = 0$ if $i \notin T$. Edmonds (7) has proved that the integrality condition are not needed explicitly here. Hence c(S) is equal to the value of the dual problem of the linear programming problem which we obtain when the integrality conditions are left out.

$$c(S) = \max \sum_{\emptyset \neq T \subset S} y_T \quad \text{subject to}$$

$$\sum_{T \subset N} y_T\, 1_T(i)(1 - 1_T(j) \leqq k(l_{ij}), \quad \text{for all } i,j \in N \cup \{0\}$$

$$y_T \geqq 0 \quad \text{for all } \emptyset \neq T \subset N.$$

Let $\hat{y}$ be an optimal solution for the dual problem which determines c(N). Define $z \in \mathbb{R}^n$ by $z_i = \sum_T 1_T(i)\hat{y}_T |T|^{-1}$. Then $z(N) = \sum_{\emptyset \neq T \subset N} \hat{y}_T = c(N)$. Let $S \in 2^N \setminus \{\emptyset\}$ be a coalition with minimum cost spanning tree T_S. For every $i \in S$ we denote the vertex which immediately precedes i on the unique path from 0 to i in T_S by p(i). Then $c(S) = \sum_{i \in S} k(l_{i\,p(i)})$ and $z(S) = \sum_{T \cap S \neq \emptyset} \hat{y}_T |T \cap S|\, |T|^{-1} \leqq \sum_{T \cap S \neq \emptyset} \hat{y}_T \leqq$

$\sum_{i \in S} \sum_{T \subset N} \hat{y}_T\, 1_T(i)(1 - 1_T(p(i))) \leqq \sum_{i \in S} k(l_{i\,p(i)}) = c(S)$ and it follows that $z \in C(c)$. The second inequality follows from the fact that a $T \subset N$ with $T \cap S \neq \emptyset$ cannot contain both i and p(i) for all $i \in T \cap S$ because then T_S would not be a spanning tree in G_S. So minimum cost spanning tree games can also be treated in the general way we described.

5. CONCLUSIONS

We have studied seven combinatorial games and we have given several for the tridimensional assignment game and the bipermutation game. The sequencing game raises another difficulty if we want to approach it in this way. In the computation of $v(N)$, we have to maximize the cost savings over all permutations of N. Because of the Birkhoff-von Neumann theorem, maximizing a linear function over the set of $n \times n$- permutation matrices is the same as maximizing the function over the set of $n \times n$-doubly stochastic matrices. In the case of the sequencing game the function that we want to maximize is only given in the points corresponding to permutation matrices. So if this function could be extended to a linear function on the set of doubly stochastic matrices there would be no problem. However, the following example shows that even in the case that the cost functions of the customers are linear this need not be possible.

EXAMPLE. Let $N = \{1,2,3\}$, $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 3$, $s_1 = 7$, $s_2 = 3$, $s_3 = 5$, $c_1(t) = 10t$, $c_2(t) = 20t$, $c_3(t) = 30t$. We consider the six permutation matrices corresponding to the permutations of N.

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Let $w(P_i)$ be the cost savings if the customers are arranged according to P_i. Then $w(P_1) = 0$, $w(P_2) = -10$, $w(P_3) = 110$, $w(P_4) = 150$, $w(P_5) = 270$, $w(P_6) = 260$. Let D be the doubly stochastic matrix defined by

$D = \frac{1}{3} P_1 + \frac{1}{3} P_4 + \frac{1}{3} P_5$ then $D = \frac{1}{3} P_2 + \frac{1}{3} P_3 + \frac{1}{3} P_6$.

It follows that for w to be extendable to a linear function on the set of 3×3-doubly stochastic matrices the following must hold :

$$\frac{1}{3} w(P_1) + \frac{1}{3} w(P_4) + \frac{1}{3} w(P_5) = \frac{1}{3} w(P_2) + \frac{1}{3} w(P_3) + \frac{1}{3} w(P_6).$$

But the left hand sight of the equation equals

$\frac{1}{3} \cdot 0 + \frac{1}{3} \cdot 150 + \frac{1}{3} \cdot 270 = 140$ and the right hand sight equals

$\frac{1}{3} \cdot -10 + \frac{1}{3} \cdot 110 + \frac{1}{3} \cdot 260 = 120$, so w cannot be extended to a linear function on the set of doubly stochastic matrices.

results for these games. For six of the seven games it has been proved that the core is not empty. For several of the games other allocation methods have also been studied. For the symmetric traveling salesman game it is still an open problem whether or not it has a non-empty core. The same holds for sequencing games with general cost functions. In the cases where the core is empty it is of course necessary to look for other reasonable allocations. But even when the core is not empty the question of how revenues or costs should be allocated is not completely answered. In general the core contains more than one element and it is not clear which core element should be chosen or even if a core element should be chosen. But again, this is not a question which could or should be answered by game theoreticians only. All they can do is study the properties of the different allocations for the different situations. It is to the people who are in need of an allocation to decide which properties they want the allocation to have and hence which allocation serves their purpose best. Especially a set of properties which uniquely determine an allocation can be very useful in this context and one should try to find such characterizing properties stated in terms which are natural for the underlying situation.

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CHAPTER XI

LINEAR OPTIMALIZATION GAMES

by Jos Potters

1. INTRODUCTION

The subject of this contribution is the theory of cooperative games with transferable utility or more precisely cooperative games arising from optimalization problems. In the literature of the last two decades there is an overwhelming amount of papers describing how optimalization situations give rise to cooperative games with transferable utility. Therefore the need for unification was experienced. Two recent papers (Dubey/Shapley (1984) and Kalai/Zemel (1982a) made an attempt to unification. They start from the same underlying idea, namely that coalitions have a set of feasible actions (or possibilities) and that they are optimizing a given function over the set of feasible actions, i.e. for each coalition S

$$w(S) = \frac{\max}{\min} \{f(x) \mid x \in \Theta(S)\}$$

where f is the function to be optimized and $\Theta(s)$ is the set of feasible actions of coalitions S. The action spaces $\Theta(s)$ are supposed to be subsets of a parameter space $\mathbb{R}^p$. In Dubey/Shapley (1984) the action spaces are described by equalities and inequalities like

$$x \geqq 0, \quad g_i(x) \leqq c_i \quad \text{and} \quad g_j(x) = c_j \quad \text{for } i \in I \text{ and } j \in J$$

and only the numbers $(c_i)_{i \in I}$ and $(c_j)_{j \in J}$ are dependent of the coalition to be considered. In fact, the constraints $c_i(c_j)$ can take only two values 0 and a value $c_i^\circ(c_j^\circ) \geqq 0$, dependent on the fact, if a coalition has access to resource i (j) or not. In Kalai/Zemel (1982a) each coalition has a set of parameters $p(S) \subset p$ under control. The other parameters are bound to vanish. This means $\Theta(S) \subset \mathbb{R}^{p(S)} \subset \mathbb{R}^p$. Both papers give sufficient conditions which the action spaces $\Theta(S)$ and the function f should satisfy in order to give rise to a cooperative game with a non-empty core.

In this paper we also endeavour an attempt to unification. In contrast with the papers mentioned before we explicitly include the possibility of the action spaces being discrete. The approach of Dubey/Shapley and Kalai/Zemel doesn't seem very suited to cover optimalization situations of this kind. On the other hand we only investigate the case of linear goal functions; in Dubey/Shapley and Kalai/Zemel the goal functions are only supposed to have a kind of concavity property.

The purpose of this chapter is to *exhibit the underlying optimalization situation* of many examples of cooperative games, to *unify the methods* to prove the existence of core elements and to *emphasize the problems* arising *when the action spaces are discrete*.

Although outside of the scope of this chapter, we mention other recent attempts to unification by Quinzii (1984), Wako (1986) and Kaneko (1982, 1983). These papers are in the field of the NTU - games (= cooperative games with non-transferable utility). But, since cooperative games with transferable utility can be understood as special cases of NTU-games, these authors have been able to cover many optimalization situations too.

2. THE MODEL AND THE METHODS

Let us consider an economic situation where n agents (persons or firms) are involved. The set of agents $N = \{1,2,\dots,n\}$ will be the *player set* of the cooperative game to be considered. Let X be the set of all imaginable actions which an agent or a coalition of agents may be able to carry out. For each coalition $S \subset N$, there is a subset $X(S) \subset X$ of actions which lie inside the possibilities of coalition S, the S-*feasible actions*. Moreover, there is an *evaluating function* $W : X \to \mathbb{R}$ which assigns to an action the monetary consequences (profits, losses or cost) of that action i.e. as far as the model covers the reality, the total change in welfare caused by the chosen action is completely described by the value of the function W.

Under these circumstances it is natural for a coalition to optimize the function W over the possible actions and we define

$$w(S) = \frac{\max}{\min} \text{ or } \frac{\sup}{\inf} \{W(x) \mid x \in X(S)\}$$

if these optimal values exist.

If each coalition S has a feasible action $x_S \in X(S)$ optimizing W, then we call the situation described above an *optimalization situation* and the associated cooperative game (N,w) an *optimalization game*. If, moreover, the set X is a subset of a finite dimensional linear vector space and W is a linear function, then we use the terms *linear optimalization situation* and *linear optimalization game*.

In the following sections many examples of cooperative games which can be understood as linear optimalization games, will pass in review. One of the principal goals of this paper is to bring more unity in the methods of proving the existence of core elements.

Let (N,w) be a cooperative game with transferable utility. The core of the game (N,w) (notation : C(w)) is defined by

$$\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = w(N) \text{ and } \sum_{i \in S} x_i \geqq (\text{or } \leqq)\ w(S) \text{ for all } S \subset N\}$$

where the $\geqq$-sign should be read if players 'like high payoffs' (profit) and the $\leqq$-sign if they 'dislike high payoffs' (losses or cost).

Essentially, there are two major ways to prove the existence of core elements.

1. *The Bondareva/Shapley approach.* A crucial role in this way to prove the existence of core elements is played by the theorem, independently proved by Bondareva (1963) and Shapley (1967).
   Theorem 2.1 (Bondareva/Shapley) : A cooperative game (N,w) has a non-empty core if and only if for every non-negative solution $\{y_S\}_{S \subset N}$

of the equation $\sum_{S\subset N} y_S e_S = e_N$ the inequality $\sum_{S\subset N} y_S w(S) \leqq$ (or $\geqq$) $w(N)$ holds. The inequality signs are opposite to the inequality signs used in the definition of the core. The vectors $e_S \in \mathbb{R}^N$ are the characteristic vectors of $S \subset N$.

For linear optimalization situations the following alternative version of theorem 2.1 will be very useful :

Theorem 2.1 (bis) : Let $\langle N, \{X(S)\}_{S \subset N}\rangle$ be a linear optimalization situation and $X(N) \subset X$ compact. Then the associated cooperative game has a non-empty core for *every linear function* W if and only if $\sum_{S\subset N} y_S X(S)$ is contained in $co(X(N))$, the convex hull of $X(N)$, for every non-negative solution $\{y_S\}_{S \subset N}$ of the equation $\sum_{S\subset N} y_S e_S = e_N$.

Remark : The advantage of theorem 2.1 (bis) compared with the original Bondareva/Shapley theorem is that only the feasible sets $X(S)$ are to be considered. A drawback is that this theorem can only be used if every linear function W yields a cooperative game with non-empty core.

Proof : We prove the maximum-version i.e. if

$$w(S) = \max\{W(x) \mid x \in X(S)\}.$$

Suppose $\sum_{S\subset N} y_S e_S = e_N$ with $y_S \geqq 0$ for all coalitions $S \subset N$. Let $x_S \in X(S)$ be an S-feasible action maximizing W over $X(S)$.
Then $\sum_{S\subset N} y_S\, w(S) = \sum_{S\subset N} y_S\, W(x_S) = W(\sum_{S\subset N} y_S x_S)$ by linearity of W.
Because $\sum_{S\subset N} y_S\, X(S) \subset co(X(N))$ and *linear* functions have the same maximum over $X(N)$ as over $co(X(N))$, we find

$$\sum_{S\subset N} y_S\, w(S) \leqq \max\{W(x) \mid x \in co(X(N))\} = w(N).$$

The core $C(w)$ is non-empty by theorem 2.1. Conversely, suppose we have a non-negative solution $\{y_S\}_{S \subset N}$ of the equation $\sum_{S\subset N} y_S e_S = e_N$ and for every $S \subset N$ (with $y_S > 0$) an S-feasible action

$x_S \in X(S)$ such that $x := \sum_{S\subset N} y_S x_S \notin co(X(N))$.
Then by a well-known separation theorem, there is a linear function W with

$$W(x) > \max\{W(y) \mid y \in co(X(N))\}.$$

The cooperative game (N,w) defined by means of this linear function W satisfies the inequalities $\sum_{S\subset N} y_S\, w(S) \geqq W(x) > w(N)$ and $C(w) = \emptyset$ by theorem 2.1 □

In applying theorem 2.1 (bis) in order to prove the existence of core elements, we have to show that

$$\sum_{S\subset N} y_S\, X(S) \subset co(X(N))$$

for all non-negative solutions of $\sum_{S\subset N} y_S e_S = e_N$. This may be very easy but can also be very unpleasant, if X(N) is discrete. In the following sections we shall in fact meet the following situation :

There is a convex subset $Y \subset X$ described by inequalities of the kind $Ax \leqq b$ and $x \geqq 0$ where A is an s×t-matrix (t = dim X) and $b \in \mathbb{R}^s$ such that the integer-valued points of Y are precisely the points of X(N). In that situation the proof of $\sum_{S\subset N} y_S\, X(S) \subset co(X(N))$ splits up into two parts :

(i) Prove $\sum_{S\subset N} y_S\, X(S) \subset Y$

and

(ii) prove that the extreme points of Y are integer-valued (and lie consequentially in X(N)).

2. *The Owen approach.* Computing the value of w(N), we have to solve a linear programming problem, P(N). Owen (1975) used the dual programm to find particular core elements. This approach has the advantage that core elements are actually found (and not only the existence of core elements is proved). Moreover the solution of the dual problem

has often an economic interpretation as 'shadow prices'. This makes the core allocation found in this way often rather convincing i.e. the players will be easier convinced that the proposed distribution of profit or cost is a fair division.

3. In one case (see section 7) there will be a directly appealing distribution of cost which appears to be a core allocation.

In the next sections the following cooperative games will pass in review :

1. Assignment Games (section 3).
2. Exchange Market Games (with linear utility of money) (section 4).
3. Production Games (section 5).
4. Coalitionally Controlled Flow Games (section 6).
5. Minimum Cost 0-Rooted Arborescence Games (section 7).
6. Traveling Salesman Games (section 8).

3. ASSIGNMENT GAMES

An assignment situation occurs when there are two types of agents mostly called sellers and buyers. The sellers initially have one or more items of an indivisible good and try to sell it. The buyers do not possess initially any good but they want to buy one or more items from the sellers. Each seller i has a minimum price c_i he wants to get for each item of his goods and each potential buyer j has a maximum price a_{ij} he is willing to pay for one item of the goods of seller i.
This means that all goods of seller i are equally appreciated by buyer j. We define $W_{ij} = a_{ij} - c_i$ for each seller i and each buyer j. If $W_{ij} < 0$ there will be no trade between seller i and buyer j; if $W_{ij} \geqq 0$ then this number gives the 'negotiation gap' between seller i and buyer j.
Let N be the set of buyers : $N = \{1,\ldots,n\}$ and $M = \{1,2,\ldots,m\}$ is the set of sellers. The player set is $N \cup M$ in this case.

3a. *A house market* (Shapley/Shubik (1972)).
In the cited paper each seller has one house for sale and each buyer

wants at most one house. For a coalition $S \subset N \cup M$ the possible trades can be described by an M×N-matrix X with zeroes and ones. If $X_{ij} = 1$ then the house of player $i \in M$ is sold to player $j \in N$. The constraints are the following :

$e_M X \leqq e_{S \cap N}$ (meaning : a house can only be sold to buyers of the coalition and they need only one house).

$X e_N \leqq e_{S \cap M}$ (meaning : a house can only be purchased from sellers of the coalition and they have only one house).

Hence we find $X = \mathbb{R}^{M \times N}$ and for each coalition $S \subset N \cup M$

$$X(S) := \{X \in Z^{M \times N} \mid e_M X \leqq e_{S \cap N} \text{ and } X e_N \leqq e_{S \cap M}\}.$$

The evaluating function W is defined by $W(X) = \sum_{i \in M} \sum_{j \in N} W_{ij} x_{ij} =: W * X$ The associated linear optimalization game assigns to a coalition $S \subset N \cup M$ the value $w(S) = \max\{W * X \mid X \in X(S)\}$. $w(S)$ is well-defined because $X(S)$ is a finite set. This cooperative game is the same as defined in Shapley/Shubik (1972), apart from the fact that we allow W_{ij} to be negative. We shall not discuss the existence of core elements in this case since we shall meet a more general situation in 3b.

3b. *General assignment games* (Potters/Tijs (1986), cf also Kaneko (1976) and Crawford/Knoer (1981)).

The more general situation we shall consider here, differs from the house markets of section 3a by the fact that sellers may have more then one house and buyers may be willing to purchase more than one house. Let $s = (s_i)_{i \in M}$ denote the initial possessions of the sellers and $d = (d_j)_{j \in N}$ the demand vector of the buyers. The meaning of the matrix $(W_{ij})_{i \in M, j \in N}$ is as before.

The possible trades within a coalition $S \subset N \cup M$ can be described by an M×N-matrix $X \in Z_+^{M \times N}$ where x_{ij} is the number of houses going from seller i to buyer j. The constraints are the following :

$$e_M X \leqq d * e_{S \cap N} \text{ and } X e_N \leqq s * e_{S \cap M}.$$

Here we use the notational convention : if p and q are vectors in $\mathbb{R}^N$, then $p * q$ is the vector in $\mathbb{R}^N$ with i-th coordinate $p_i q_i$ for all $i \in N$. Hence,

$$X(S) = \{X \in Z_+^{M\times N} \mid e_M X \leq d * e_{S \cap N} \text{ and } X e_N \leq s * e_{S \cap M}\}.$$

The evaluating function W is as before and $w(S) = \max\{W(X) \mid X \in X(S)\}$. Note that $d = e_N$ and $s = e_M$ gives the house market of 3a.

Theorem 3.1 : General assignment games have non-empty cores.

Proof : (The Bondareva/Shapley approach) : Let $\{y_S\}_{S \subset N \cup M}$ be a non-negative solution of $\sum_{S\subset N \cup M} y_S e_S = e_{N \cup M}$ and

$$Y = \{X \in \mathbb{R}_+^{M\times N} \mid e_M X \leq d \ (= d * e_N) \text{ and } X e_N \leq s \ (= s * e_M)\}.$$

The integer-valued points of Y are the points of $X(N \cup M)$.

Let $x = \sum_{S\subset N \cup M} y_S X_S \in \sum_{S\subset N \cup M} y_S \, X(S)$ with $X_S \in X(S)$ for each $S \subset N \cup M$.

Then $e_M X = \sum_{S\subset M \cup N} y_S \, (e_M X_S) \leq \sum_{S\subset M \cup N} y_S \, d * e_{S \cap N} = d * \sum_{S\subset N \cup M} y_S e_{S \cap N} =$ $d * e_N = d$ and analogously $X e_N \leq s$. Hence, $\sum_{S\subset N \cup M} y_S \, X(S) \subset Y$.

Further, we have to prove that the extreme points of Y are integer-valued. Because we need this result in section 4 too, we formulate the proposition :

Proposition 3.2 (cf Brualdi/Gibson (1976)) : The extreme points of the set

$$Y := \{X \in \mathbb{R}_+^{M\times N} \mid e_M X \leq d \text{ and } X e_N \leq s\}$$

are integer-valued if $d \in Z_+^N$ and $s \in Z_+^M$.

Proof : In fact we shall prove that (also if s and d are not integer-valued) all extreme points of Y are generated by the following procedure :

(i) Take a subset $A \subset M\times N$ and order the elements of A linearly

$$\alpha(1) < \alpha(2) < \ldots < \alpha(p)$$

(ii) Start with $X = 0$ and change the entries of A successively according the given order as follows :

$$x_{\alpha(k)} = \min(s_{i(k)}, d_{j(k)}) \text{ if } \alpha(k) = (i(k), j(k))$$
$$s_{i(k)} = s_{i(k)} - x_{\alpha(k)} \text{ and } d_{j(k)} = d_{j(k)} - x_{\alpha(k)}.$$

Note that every time that $x_{\alpha(k)} > 0$ the number of completed rows and columns (i.e. rows and columns with maximal row or column sum) increases with at least one (and at most two in the exceptional case that $x_{\alpha(k)} = s_{i(k)} = d_{j(k)}$).

Let X be an extreme point of Y and $A = \{\alpha \mid x_\alpha > 0\} \subset M \times N$.
We have to find a linear order of A such that the procedure described above, generates X. The proof proceeds by induction on $|M \times N|$.
For $|M| = |N| = 1$ the set $Y = \{x \in \mathbb{R}_+ \mid x \leq \min(s,d)\}$ and the extreme points are $x = 0$ and $x = \min(s,d)$ generated by $A = \emptyset$ and $A = M \times N = \{(1,1)\}$.
In order to complete the induction we have to find an entry $\alpha \in A$ such that

$$x_\alpha = s_{i(\alpha)} \text{ or } x_\alpha = d_{j(\alpha)} \text{ where } \alpha = (i(\alpha), j(\alpha)).$$

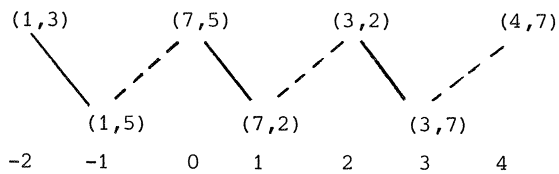
We find such an entry by constructing what we shall call a maximal i-j-string of A.
A *i-j-string* of A is a string $\alpha_q, \alpha_{q+1}, \ldots, \alpha_0, \ldots, \alpha_p$ of elements of A with the following properties :

(i) $i(\alpha_k) = i(\alpha_{k+1})$ if k is even; $j(\alpha_k) = j(\alpha_{k+1})$ if k is odd.

(ii) the entries $\alpha_q, \ldots, \alpha_p$ are all different.

(iii) each row and each column index occur at most twice, immediately after each other.

An i-j-string may, for example, look like this



An i-j-string is called *maximal* if it cannot be extended conserving the properties (i), (ii) and (iii). Since a one-element string $\alpha_0 \in A$ is an i-j-string, there are maximal i-j-strings too. Let $\alpha_q,\ldots,\alpha_p$ be a maximal i-j-string in A. Let us suppose that q and p are even. The proof is not essentially different if p or q is odd. There are two possible reasons why the string $\alpha_q,\ldots,\alpha_p$ cannot be extended with α_{q-1} :

(a) There is no entry $\alpha \in A$ with $\alpha \neq \alpha_q$ and $j(\alpha) = j(\alpha_q)$

or

(b) there is an entry $\alpha \in A$ with $\alpha \neq \alpha_q$ but $i(\alpha)$ occurs already among the row indices.

We consider the second case first. Let s be the smallest index with $i(\alpha_s) = i(\alpha)$. Then s is even and $\alpha = \alpha_{q-1},\alpha_q,\ldots,\alpha_s$ is an i-j-cycle of even length. It may be clear what is meant with i-j-cycle. We can do the same analysis at the endpoint α_p of the string and we find :

(A) an i-j-cycle of even length

of

(B) an i-j-string $\alpha_q,\alpha_{q-1},\ldots,\alpha_p$ such that the column of α_q and the row of α_p do not contain other elements of A.

In both cases (A) and (B) we define an M×N-matrix E by :

$E_\alpha = 1$ if α occurs in the i-j-cycle or string with even index.

$E_\alpha = -1$ if α occurs in the cycle or string with odd index.

$E_\alpha = 0$ if α does not occur in cycle or string.

In case (A) the matrix E has all row and column sums zero and $x \pm \varepsilon E \in y$ if $0 < \varepsilon \leqq \min\{x_\alpha \mid \alpha$ occurs in the cycle$\}$. This is in contradiction with the extremality of X.

In case (B) the matrix E has all row and column sums zero except the column sum of α_q and the row sum of α_p. If the $j(\alpha_q)$-th column sum of X *and* the $i(\alpha_p)$-th row sum of X are not maximal, then $X \pm \varepsilon E \in Y$, if $0 < \varepsilon \leqq \min\{x_\alpha \mid \alpha$ in the i-j-string; $d_{j(\alpha_q)} - x_{\alpha_q},\ s_{i(\alpha_p)} - x_{\alpha_p}\}$.

But, X is extreme point of Y. Hence, $x_{\alpha(q)} = d_{j(\alpha_q)}$ or $x_{\alpha(p)} = s_{i(\alpha_p)}$.

We assume, without loss of generality, that $x_\alpha = s_{i(\alpha)} \geqq 0$ for some entry $\alpha \in A$. Let $\bar{X}$ be the $\bar{M}\times N$-matrix obtained by skipping the $i(\alpha)$-th row. $\bar{M} = M\setminus\{i(\alpha)\}$, $\bar{s} = (s_1,\ldots,\hat{s}_{i(\alpha)},\ldots,s_m) \in \mathbb{R}_+^{\bar{M}}$ and $\bar{d} = d - x_\alpha e_{j(\alpha)} \in \mathbb{R}_+^N$. Then

$$\bar{X} \in \bar{Y} = \{X \in \mathbb{R}_+^{\bar{M}\times N} \mid e_{\bar{M}}\, X \leqq \bar{d} \text{ and } X\, e_N \leqq \bar{s}\}.$$

$\bar{X}$ is even an extreme point of $\bar{Y}$. For, if $\bar{X} = \frac{1}{2}(\bar{X}_1 + \bar{X}_2)$ with $\bar{X}_1 \neq \bar{X}_2$ and $\bar{X}_1, \bar{X}_2 \in \bar{Y}$, then $X = \frac{1}{2}(X_1 + X_2)$ where X_i is the extension of $\bar{X}_i$ with the $i(\alpha)$-th row of X. Furthermore, $X_1 \neq X_2$ and $X_1, X_2 \in Y$. This contradicts the extremality of X.

By the induction hypothesis $\bar{X}$ can be obtained by the algorithm above from an ordering of $\bar{A} = \{\alpha \in \bar{M}\times N \mid \bar{x}_\alpha\ (= x_\alpha) > 0\}$. Note that $A = \bar{A} \cup \{\alpha\}$. Then we order the elements of A by taking α first, followed by the elements of $\bar{A}$ in the order which generated $\bar{X}$. This order of A yields X as can be easily seen. Since we do not need the converse part of the proposition, we skip this part of the proof. If s and d are integer-valued, the algorithm only produces integer entries. □

Before closing this section, we will devote a few words to the model of a house market introduced by Kaneko (1976). In this model we have also two types of players, sellers and buyers. These two types are distinguished by their initial endowments and their utility functions.

The initial endowment of a seller i is an integer-valued vector $w_i = (w_{i,1},\ldots,w_{i,k}) \in Z_+^k$. Seller i has $w_{i,s}$ houses of type s for sale. The initial endowment of a buyer j is empty $(0 \in Z_+^k)$.

The utility function $u_i : Z_+^k \to \mathbb{R}$ of a seller i has the following properties

(1) $u_i(a_1,\ldots,a_k) = \sum_{s=1}^{k} u_i(a_s e_s)$.

(2) the functions $u_{is} : Z_+ \to \mathbb{R}$ defined by $u_{is}(a) = u_i(a e_s)$ are non-decreasing and have non-increasing marginals for $s = 1,\ldots,k$.

(3) $u_{is}(a) = u_{is}(w_{i,s})$ if $a \geqq w_{i,s}$ and $1 \leqq s \leqq k$.

Buyer j's utility function has the properties

(1) $u_j(a_1,\dots,a_k) = \max_{s=1,\dots,k} u_j(a_s e_s)$

and

(2) $u_{js}(0) \leqq u_{js}(1) = u_{js}(a)$ if $a \geqq 1$ and $1 \leqq s \leqq k$.

Kaneko defines a cooperative game with side payments by

$$v(\ ) = \max\{ \sum_{i\in S\cap M} u_i(x_i) + \sum_{j\in S\cap N} u_j(x_j) \mid \sum_{i\in S\cap M} x_i + \sum_{j\in S\cap N} x_j \leqq \sum_{i\in S\cap M} w_i \}.$$

The following trick (also considered by Kaneko) changes this house market into a house market of the Shapley/Shubik type. Think of seller i as a firm with $\Sigma_{s=1}^{k} w_{i,s}$ agents, indexed by (s,f) where $1 \leqq s \leqq k$ and $1 \leqq f \leqq w_{i,s}$.

Agent i(s,f) has the f-th house of type s for sale and has a minimum price $u_i(fe_s) - u_i((f-1)e_s)$, the marginal utility of the f-th house of type s. Buyer j's maximum price is $u_j(e_s)$. Core elements of Kaneko's game can be obtained by summing up the payoffs of all agents of one selling firm i of a core allocation in the Shapley/Shubik market.

4. EXCHANGE MARKET GAMES

In an exchange market the same agents are selling and buying. Each agent $j \in N$ has an initial endowment with indivisible goods (say houses) of different type

$$Q_j = (Q_{1j}, Q_{2j}, \dots, Q_{mj}).$$

The set $M = \{1,\dots,m\}$ is the set of house types. By exchanging their houses under monetary compensation, the agents try to improve their situation. The M×N-matrix $W = (W_{ij})_{i \in M, j \in N}$ is given as before and expresses the appreciation of agent j for a house of type i. Further the vector $d = (d_j)_{j \in N} \in Z_+^N$ is the demand vector i.e. d_j is the number of houses that agent j finally wants to possess.

A special case of an exchange market is a permutation situation

4a. *Permutation games* (Tijs et al. (1984), Curiel/Tijs (1985)).

In a permutation situation all agents possess one house and need also one house. I.e. the initial endowments are $e_j \in Z_+^N$, $N = M$ and the demand vector d is $e_N = (1,1,...,1)$.
The feasible actions of a coalition $S \subset N$ are the permutations of s i.e.

$$X(S) = \{X \in Z_+^{N\times N} \mid e_N X = e_S \text{ and } X e_N = e_S\}.$$

The evaluating function W is defined by $W(X) = W * X$ and the permutation game is the linear optimalization game

$$w(S) = \max\{W * X \mid X \in X(S)\}.$$

The fact that in Tijs et al. (1984) a cost game is considered does not influence the essential features of the situation since cost games can easily be transformed in cost-saving (i.e. profit) games. But let us proceed in a more general situation.

4b. *Exchange market games* (Potters/Tijs (1986)).

The tradings feasible for coalition S can be described by an M×N-matrix $X \in Z_+^{M\times N}$ where x_{ij} denotes the number of houses of type i which agent $j \in N$ finally possesses (and appreciates).
The following constraints should be satisfied $e_M X \leqq d * e_s$ ($d * e_s$ is the demand vector of coalition S), $X e_N \leqq Q e_s$ ($Q e_s$ is the supply vector of coalition S).
Hence,

$$X(S) = \{X \in Z_+^{M\times N} \mid e_M X \leqq d * e_S \text{ and } X c_N \leqq Q e_S\}$$

and the associated optimalization game is defined by

$$w(S) = \max\{W * X \mid X \in X(S)\}.$$

In Potters/Tijs (1986) the following theorem has been proved :
Theorem 4.1 : Exchange market games have non-empty cores.

Proof : Suppose, $\{y_S\}_{S \subset N}$ is a non-negative solution of $\sum_{S \subset N} y_S e_S = e_N$. Then for $X = \sum_{S \subset N} y_S X_S$ with $X_S \in X(S)$ for all $S \subset N$, we have $e_M X = \sum_{S \subset N} y_S (e_M X_S) \leqq \sum_{S \subset N} y_S\ d * e_S = d * \sum_{S \subset N} y_S e_S = d * e_N = d$

and

$X\, e_N = \sum_{S \subset N} y_S (X_S e_N) \leqq \sum_{S \subset N} y_S\ Q\, e_S = Q\, e_N$

i.e. $\sum_{S \subset N} y_S\, X(S) \subset Y := \{X \in \mathbb{R}_+^{M \times N} \mid e_M X \leqq d \text{ and } X\, e_N \leqq Q\, e_N\}$.

The integer-valued points of Y are precisely the points of X(N) and the extreme points of Y are integer-valued by proposition 3.1. Hence the theorem follows from the alternative version of the Bondareva/Shapley theorem. □

For permutation games we get $\sum_{S \subset N} y_S\, X(S) \subset Y$ where

$Y = \{X \in \mathbb{R}_+^{N \times N} \mid e_N X = e_N \text{ and } X\, e_N = e_N\}$.

The extreme points of Y, the set of double stochastic matrices, are the points of X(N) by the theorem of Birkhoff/Von Neumann.

5. PRODUCTION GAMES (Owen (1985), Granot (1986)).

Suppose, there is a finite set of producers $N = \{1,\ldots,n\}$ and each coalition $S \subset N$ can dispose of certain amounts of raw materials, resources. More precisely for each coalition S and each kind of resource j, $b(S)_j \in \mathbb{R}_+$ is the amount of resource j which coalition S has at its disposal. From the resources they have at their disposal, the agents in a coalition can produce certain quantities of products. If P is the set of products to be considered, a production plan is given by a vector $x \in \mathbb{R}_+^P$. The connection between resources and products is given by a $P{\times}R$-matrix $A \geqq 0$, the production matrix. The entry A_{kj} is the amount of resource j needed for the production of one unit of product k. We assume that all producers use identical production processes i.e. have the same production matrix.

Furthermore, there is a vector $c \in \mathbb{R}^P$ where c_k is the net profit which can be earned by producing and selling one unit of product k.
The possibilities of a coalition S are given by a production vector $x \in \mathbb{R}^P_+$ under the constraints $xA \leqq b(S)$ (there have to be enough raw materials available). The evaluating function W is the inner product with c, the total net profit.
Hence,

$$X(S) = \{x \in \mathbb{R}^P_+ \mid xA \leqq b(S)\} \text{ and}$$
$$w(S) = \max\{\langle c,x \rangle \mid x \in X(S)\}.$$

Remark : In Owen (1975) the producers have a resource vector $b_i \in \mathbb{R}^R_+$ at their disposal and $b(S) = \sum_{i\in S} b_i$. Granot (1986) investigates the more general situation described above.

Theorem 5.1 (Granot (1986), Owen (1975)). If the 'resource games' $S \to b(S)_j$ have non-empty cores for all $j \in R$, then the production game has also a non-empty core.

Proof : Let $\{y_S\}_{S\subset N}$ be a non-negative solution of $\sum_{S\subset N} y_S e_S = e_N$ and let $x = \sum_{S\subset N} y_S x_S$ with $x_S \in X(S)$. Then $x \in \mathbb{R}^P_+$ and

$xA = \sum_{S\subset N} y_S\ (x_S A) \leqq \sum_{S\subset N} y_S\ b(S)$. Because $\sum_{S\subset N} y_S\ b(S) \leqq b(N)$ (the resource games have non-empty cores) we find $xA \leqq b(N)$ and $x \in X(N)$. The core is non-empty by theorem 2.1 (bis). □

We give also an alternative proof of theorem 5.1 using the duality theory of linear programming.
Proof : (Owen approach) : Computing the value w(S), we have to solve the L.P. problem.

$$P(S): \quad \begin{array}{l} \text{Maximize } \langle c,x \rangle \text{ under the constraints} \\ xA \leqq b(S) \text{ and } x \geqq 0. \end{array}$$

The dual problem is

$$D(S) \quad \begin{array}{l} \text{Minimize } \langle y, b(S) \rangle \text{ under the constraints} \\ Ay \geqq c \text{ and } y \geqq 0. \end{array}$$

Note that both L.P. problems are feasible ($A \geqq 0$ is important here) and further that the feasibility domain of $D(S)$ is independent of S.

Consider the L.P. problem $P(N)$ and $D(N)$. We have by duality theory

$$w(N) = \min\{\langle y, b(N) \rangle \mid Ay \geqq c \text{ and } y \geqq 0\}.$$

Let $\hat{y} \in \mathbb{R}^R_+$ be an optimal solution of $D(N)$ and $a_j \in \mathbb{R}^N$ a core element of the resource games $S \to b(S)_j$ for $j = 1,\dots,r$.
Define $u_i := \sum_{j \in R} \hat{y}_j . a_{ji}$ for $i = 1,\dots,n$. Then $u = (u_i)_{i \in N}$ in a core element. We prove this directly :

$\sum_{i \in N} u_i = \sum_{j \in R} \hat{y}_i \sum_{i \in N} a_{ji} = \sum_{j \in R} \hat{y}_j . b(N)_j$, since a_j is an efficient allocation in the game $S \to b_j(S)$. Then $\sum_{i \in N} u_i = \langle \hat{y}, b(N) \rangle = w(N)$.
Furthermore, $\sum_{i \in S} u_i = \sum_{j \in R} \hat{y}_j \sum_{i \in S} a_{ji} \geqq \sum_{j \in R} \hat{y}_j . b(S)_j = \langle \hat{y}, b(S) \rangle$.

Since $\hat{y}$ is feasible in $D(S)$, we find

$$\sum_{i \in S} u_i \geqq \min\{\langle y, b(S) \rangle \mid Ay \geqq c \text{ and } y \geqq 0\} = w(S) \text{ by duality.} \qquad \square$$

Remark 1 : If we take the vector $\hat{y} \in \mathbb{R}^R_+$ as shadow prices for the resources pro unit, then u_i is the value of the resources assigned to producer i under the core allocations $(a_1,\dots,a_r)$.

Remark 2 : Assignment games as well as exchange market games can be understood as production games. In both types of games the 'products' are trades (i,j) and the resources are $N \cup M$. The production of one unit (i,j) demands one unit i and one unit j. The price vector is the matrix $W = (w_{ij})$.

For *assignment games* the player set is $N \cup M$, the resource vector of seller i is $b_i = (0,\dots,s_i,\dots,0.,0,\dots0)$ and the resource vector of buyer j is

$b_j = (0,\dots 0;0,\dots d_j,\dots 0)$.

A coalition $S \subset N \cup M$ has the resource vector $b(S) = \sum_{i \in S \cap M} b_i + \sum_{j \in S \cap N} b_j$.

The production constraints are : $\sum_{i \in M} x_{ij} \leqq d_j \, e_{S \cap N, j} = (d * e_{S \cap N})_j$

and $\sum_{j \in N} x_{ij} \leqq (s * e_{S \cap M})_i$ (cf section 3b).

For *exchange market games* the player set is N and the resource vector of agent $j \in N$ is $b_j = (Q_{1j},\dots,Q_{mj};0,\dots,d_j,\dots 0)$.

The production constraints are

$$\sum_{i \in M} x_{ij} \leqq (d * e_S)_j \text{ and } \sum_{j \in N} x_{ij} \leqq \sum_{j \in S} Q_{ij} = (Q\, e_S)_i \quad \text{(cf section 4b).}$$

Remark 3 : Let us consider the shadow prices in both types of games.

Assignment games : Shadow prices are optimal solutions of $D(N \cup M)$ i.e. minimize $\langle y,s \rangle + \langle z,d \rangle$ under the constraints $y \in \mathbb{R}_+^M$, $z \in \mathbb{R}_+^N$ and $y_i + z_j \geqq W_{ij}$ for all $i \in M$ and $j \in N$.

Then the core element generated by these shadow prices is

$$((s_i \hat{y}_i)_{i \in M}, (d_j \hat{z}_j)_{j \in N}) \in \mathbb{R}^{M \cup N}.$$

If $\hat{x}$ is an optimal assignment, then $\hat{x}_{ij} > 0$ implies $\hat{y}_i + \hat{z}_j = W_{ij}$ by complementary slackness. Recalling that $W_{ij} = a_{ij} - c_i$, we find that if seller i and buyer j negotiate succesfully $(\hat{x}_{ij} > 0)$ then $\hat{y}_i + c_i = a_{ij} - \hat{z}_j$. This is the price paid by player j to player i.

For *exchange markets* the shadow prices are optimal solutions of $D(N)$ i.e. minimize $\langle y, Q\, e_N \rangle + \langle z,d \rangle$ under the constraints $y \in \mathbb{R}_+^M$, $z \in \mathbb{R}_+^N$ and $y_i + z_j \geqq W_{ij}$ for all $i \in M$ and $j \in N$.

The core allocation generated by these shadow prices is $u = (u_j)_{j \in N}$ with $u_j = \sum_{i \in M} \hat{y}_i Q_{ij} + \hat{z}_j d_j$ for all $j \in N$.

Remark 4 : In Potters/Tijs (1986) pooling situations are investigated. These are situations which can be described just as well as an assignment game as an exchange market game. The initial endowments are the option rights to buy certain amounts of products from the sellers.

The Owen approach works very nicely in such a situation because the dual L.P. problems D(N $\cup$ M) and D(N) are identical if we put $s = Q e_N$. Hence the same shadow prices prevail and the core elements generated by these shadow prices are connected by the relation

$$u_j = \sum_{i \in M} (\hat{y}_i s_i) \frac{Q_{ij}}{\sum_{j \in N} Q_{ij}} + (\hat{z}_j d_j) \text{ for all } j \in N.$$

Note that $(\sum_{j \in N} Q_{ij})^{-1} Q_{ij}$ is the fraction of $s_i = \sum_{j \in N} Q_{ij}$ of which player j has the option rights.

6. FLOW GAMES WITH COALITIONAL CONTROL (Kalai/Zemel (1982), Curiel/Tijs (1987))

Let P be a finite point set with two particular points a and b. Suppose, the connections between different points of P are controlled by coalitions of a player set N in the following way.

For each coalition S there is a capacity map $c(S) : P \times P \to \mathbb{R}_+$, expressing to what extent the connections between points of P are allowed to be used by coalition S. Hence, for each coalition $S \subset N$, we have a network with capacities (P, c(S)).

A *flow* from a to b is a $P \times P$-matrix $X \in \mathbb{R}_+^{P \times P}$ such that

$$\sum_{i \in P} x_{ij} = \sum_{i \in P} x_{ji} \qquad \text{for all } j \in P \setminus \{a,b\}.$$

A flow is *feasible* for coalition S if, moreover, $0 \leqq x_{ij} \leqq c(S)(i,j)$ for all $(i,j) \in P \times P$.

Then it is easy to prove that $\sum_{i \in P} x_{ai} - \sum_{i \in P} x_{ia} = \sum_{i \in P} x_{i,b} - \sum_{i \in P} x_{b,i}$.

This is the value of the flow X.

We define a flow game with coalitional control as a linear optimalization game with

$$X(S) = \{X \in \mathbb{R}_+^{P\times P} \mid e_P\, Xe_i = e_i\, Xe_P \text{ for } i \neq a,b \text{ and } x_{ij} \leqq c(S)(i,j) \text{ for all } (i,j) \in P\times P\}$$

and the evaluating function $W(X) = e_a\, X\, e_P - e_P\, X\, e_a$, the value of the flow.

We prove the following theorem :

Theorem 6.1 : A flow game with coalitional control has a non-empty core if the capacity games $S \to c(S)(i,j)$ have a non-empty core for all pairs $(i,j) \in P\times P$.

Proof : We proof that $\sum_{S\subset N} y_S\, X(S) \subset X(N)$ for all non-negative solutions of $\sum y_S e_S = e_N$. Let $X = \sum_{S\subset N} y_S X^S$ with $X^S \in X(S)$ for all coalitions S (with $y_S > 0$).

Then $e_i X\, e_P = \sum_{S\subset N} y_S\, (e_i\, X^S\, e_P) = \sum_{S\subset N} y_S\, (e_P\, X^S\, e_i)$. (if $i \in P \setminus \{a,b\}$) =

$= e_P\, X\, e_i$. Furthermore,

$x_{ij} = \sum_{S\subset N} y_S\, (X^S)_{ij} \leqq \sum_{S\subset N} y_S\, c(S)(i,j) \leqq c(N)(i,j)$ because $\{c(S)(i,j)\}_{S\subset N}$ satisfies the Bondareva/Shapley theorem. So $X \in X(N)$ and the non-emptiness of the core follows from theorem 2.1 (bis). □

An alternative proof uses the max-flow-min-cut theorem of Ford/Fulkerson (1956).

Proof : Let $\hat{X}$ be a maximal flow in the network $(P,c(N))$. Then by the theorem of Ford/Fulkerson there is a cut $(Q,P\backslash Q)$ with $a \in Q$ and $b \notin Q$ such that

$$\sum_{i\in Q} \sum_{j\notin Q} c(N)(i,j) = w(N) \text{ (the value of the maximal flow } \hat{X}).$$

For all pairs $(i,j) \in Q \times P\backslash Q$ we take a core element $a_{ij} \in \mathbb{R}^N$ of the capacity game $S \to c(S)(i,j)$. Define $u = (u_k)_{k \in N}$ by

$$u_k = \sum_{i\in Q} \sum_{j\notin Q} a_{ij}(k).$$

Then $u \in C(w)$. For, $u(N) = \sum_{i\in Q} \sum_{j\notin Q} a_{ij}(N) = \sum_{i\in Q} \sum_{j\notin Q} c(N)(i,j) = w(N)$

and $u(S) = \sum_{i\in Q} \sum_{j\notin Q} a_{ij}(S) \geqq \sum_{i\in Q} \sum_{j\notin Q} c(S)(i,j)$.

Because $(Q,P\setminus Q)$ is a cut in network $(P,c(S))$ too, we find

$$u(S) \geqq \sum_{i\in Q} \sum_{j\notin Q} c(S)(i,j) \geqq \min_{Q'}\{ \sum_{i\in Q'} \sum_{j\notin Q'} c(S)(i,j) \mid a \in Q' \subset P\setminus\{b\}\} =$$

$w(S)$ by Ford/Fulkerson. □

7. MINIMUM COST 0-ROOTED ARBORESCENCE GAMES (Granot/Huberman (1981)).

Consider the situation of n cities which want to be connected with a water resource 0. The cities decide to cooperate and to build a common pipe-line system connecting each city with the water resource. Let C be a $N_0 \times N$-matrix of which the entries C_{ij} denote the cost of a pipe-line from city i to city j. We do not assume that the matrix C is symmetric (e.g. the connection from i to j may require more water-pumps than a connection in the opposite direction). The feasible actions of a coalition S are all pipe-line systems connecting the cities of S with the water resource only using the cities of S as intermediate points. So, we are looking for a directed graph with nodes $S_0 = S \cup \{0\}$ and the following properties :

1. There is exactly one edge e_i pointing to a city $i \in S$.
2. For every city $i \in S$ there is a directed path from 0 to i.

The second condition can be replaced by

2'. For every non-empty set $T \subset S$ there is at least one edge pointing from outside of T to a point of T.

If we represent the graph on S_0 by its incidence matrix we find the conditions

$$e_{S_0} X = e_S \text{ and } e_{N_0\setminus T} X e_T \geqq 1 \text{ if } T \subset N \text{ and } T \cap S \neq 0, \text{and } e_{N\setminus S} X = 0.$$

Hence we define

$$X(S) = \{X \in Z_+^{N_0 \times N} \mid e_{S_0} X = e_S, e_{N\setminus S} X = 0 \text{ and } e_{N_0\setminus T} X e_T \geqq 1 \text{ for all } T \subset N \text{ with } T \cap S \neq \emptyset\}.$$

It is easy to see that $x_{ij} = 0$ or 1 if $X \in X(S)$. Furthermore, for each $i \in S$ there is exactly one city $j \in S_0$ such that $x_{ji} = 1$. Hence for each $i \in S$ we can define a sequence $i_t, i_{t-1}, \ldots, i_0 = i$ such that $x_{i_j i_{j-1}} = 1$ for $j = 1,2,..t$. We can not get into a cycle $i_t = i_0, i_{t-1},..i_0$ for in that case $e_{N_0\setminus T} X e_T = 0$ if we take $T = \{i_0, i_1 \ldots, i_{t-1}\} \subset S$. Hence we find a path from $i_t = 0$ to i.

The optimalization game belonging to this situation is defined by

$$w(S) = \min\{C * X \mid X \in X(S)\}.$$

We call this type of game a Minimum Cost 0-Rooted Arborescence Game or MCA-game for short. We can immediately give a core element in MCA-games and therefore :

Theorem 7.1 : (cf Granot/Huberman (1981)) : MCA-games have non-empty cores.

Proof : Let $\hat{X} \in X(N)$ be a 0-rooted arborescence with minimum cost. Let, for each $i \in N$, $p(i)$ be the unique element in N_0 with $x_{p(i),i} = 1$. Define $u_i = c_{p(i),i}$ for all $i \in N$. Then $u = (u_i)_{i \in N}$ is a core allocation. Clearly $\sum_{i \in N} u_i := \sum_{i \in N} c_{p(i),i}$ is the total cost $C * \hat{X} = w(N)$.

Let S be a non-empty coalition and $X_S \in X(S)$ a 0-rooted arborescence for the cities of S. We extend the 0-rooted arborescence X_S by adding the arcs $(p(i),i)$ of $\hat{X}$ if $i \notin S$. Then we only have to check the condition $e_{N_0\setminus T} X e_T \geq 1$ if $T \subset N$, non-empty. If $T \cap S \neq \emptyset$, then there is an arc of X_S pointing from the outside of $T \cap S$ to a point of $T \cap S$. This arc is also in the extension of X_S.

If $T \cap S = \emptyset$, there is an arc of $\hat{X}$ from $N_0\setminus T$ to a point of T. The end-point is a point of T and not a point of S. So, this arc also lies in the extension of $\hat{X}_S$. Hence we find $C * X_S + \sum_{i \in N\setminus S} u_i \geq w(N) = \sum_{i \in N} u_i$.

Therefore, $\sum_{i \in S} u_i \leq C * X_S$ for all 0-rooted arborescences in S_0.

$\sum_{i \in S} u_i \leq w(S)$ and $u = (u_i)_{i \in N}$ is a core element. □

We also try the Bondareva/Shapley approach.

Proof : Let $\{y_S\}_{S \subset N}$ be a non-negative solution of the equation $\sum_{S \subset N} y_S e_S = e_N$ and $X = \sum_{S \subset N} y_S X_S$ with $X_S \in X(S)$.

Then $e_{N_0\backslash T} X e_T = \sum_{S\subset N} y_S e_{N_0\backslash T} X e_T \geq \sum_{S\subset N, T\cap S \neq \emptyset} y_S \geq 1$ if $T \neq \emptyset$.

$$e_{N_0} X = \sum_{S\subset N} y_S e_{N_0} x_S = \sum_{S\subset N} y_S (e_{S_0} x_S) = \sum y_S e_S = e_N.$$

Hence, we find $\sum_{S\subset N} y_S X(S) \subset Y = \{X \in \mathbb{R}_+^{N_0\times N} \mid e_{N_0} X = e_N$ and $e_{N_0\backslash T} X e_T \geq 1$ if $T \neq \emptyset\}$.

We have to prove that the extreme points of Y are integer-valued.
A proof of this fact can be found in Edmonds (1967). Then theorem 2.1 (bis) gives the non-emptiness of the core. □

Remark : If we want to prove directly, that $Y = co(X(N))$, we have to find with each element $X \in Y$ an element $X_0 \in X(N)$ such that

(i) $(x_0)_{ij} = 1$ implies $x_{ij} > 0$

(ii) if $e_{N_0\backslash T} X e_T = 1$ for some $T \subset N$, then $e_{N_0\backslash T} X_0 e_T = 1$ too.

If this can be done, then $(1-\varepsilon)^{-1}(X - \varepsilon X_0) \in Y$ if ε is chosen not too large and, in fact, we can choose $\varepsilon > 0$ in such a way that $(1-\varepsilon)^{-1}(X-\varepsilon X_0)$ has more entries zero than X or more equalities of the kind $e_{N_0\backslash T} X e_T = 1$. Note that for a 0-rooted arborescence, $e_{N_0\backslash T} X_0 e_T = 1$ means that T is connected in the graph X_0.

8. TRAVELING SALESMAN GAMES (Potters/Curiel/Tijs (1987)).

A speaker with residence t_0 is invited by n universities $t_1,\ldots,t_n$ for a lecture. Therefore he plans a round-trip along the universities with minimal travel cost. The total travel cost has to be paid by the universities visited. Let $(c_{ij})_{i,j \in N_0}$ be the cost matrix i.e. c_{ij} denotes the travel cost on the traject t_i-t_j. We want to find a distribution of the total cost among the universities such that no coalition of universities has an incentive to split off and to invite the speaker for their own universities only. This means : we look for core elements of the game (N,w) defined by

$$w(S) = \min\{\sum_{i\in S_0} c_{i,\pi(i)} \mid \pi \text{ is a cyclic permutation of } S_0\}$$

where $S_0 = S \cup \{0\}$.

Introducing permutation matrix X_π for (cyclic) permutations π, we get

$$w(S) = \min\{C * X \mid X \in X(S)\}$$

where

$$X(S) = \{ \begin{array}{l} X \in Z_+^{N_0 \times N_0} \mid e_{N_0} X = e_{S_0},\ X e_{N_0} = e_{S_0} \text{ and } e_{N_0 \setminus T}\, X\, e_T \geqq 1 \\ \text{for all non-empty } T \subset N_0,\ T \cap S_0 \neq \emptyset,\ T \not\supset S_0. \end{array} \}$$

Note that the inequalities $e_{N_0 \setminus T}\, X\, e_T \geqq 1$ exclude non-cyclic S_0-permutations.

The results in this section will be negative for the most part.

1. The Bondareva/Shapley approach fails.

If $\Sigma\, y_S\, e_S = e_N$ and $X_S \in X(S)$ for all $S \subset N$ (with $y_S > 0$), then

$$e_{N_0} \Big(\sum_{S \subset N} y_S X_S \Big) = \sum_{S \subset N} y_S (e_{N_0} X_S) = \sum_{S \subset N} y_S\, e_{S_0} = e_{N_0} + \Big(\sum_{S \subset N} y_S - 1 \Big) e_0 \neq e_{N_0}$$

if $\sum_{S \subset N} y_S \neq 1$. But this is true unless we have the trivial equality $1 \,.\, e_N = e_N$. We find the same outcome for $\big(\sum_{S \subset N} y_S X_S \big) e_{N_0}$. This means that $\sum_{S \subset N} y_S\ X(S) \not\subset co(X(N))$.

The cost matrix should have special properties to make the core non-empty.

2. Let C be any cost matrix and disturb the entries (0,j) of C only

$$\tilde{c}_{0,j} = c_{0,j} + \alpha \quad \text{where } \alpha \in \mathbb{R} \text{ and } j \neq 0.$$

$$\tilde{c}_{i,j} = c_{i,j} \quad \text{if } i > 0.$$

Then we find

$$\tilde{w}(s) = \min\{ \sum_{i \in S_0} \tilde{c}_{i,\pi(i)} \mid \pi \text{ is cyclic in } S_0\} =$$

$$\min\{ (\sum_{i \in S_0} c_{i,\pi(i)}) + \alpha)\ \pi \text{ is cylcic in } S_0\} =$$

$$w(S) + \alpha.$$

For a non-negative solution $\{y_S\}_{S \subset N}$ of $\sum_{S \subset N} y_S\, e_S = e_N$, not equal to the trivial solution $1.\ e_N = e_N$, we obtain

$\sum_{S \subset N} y_S\, \tilde{w}(S) = \sum_{S \subset N} y_S\, w(S) + \sum_{S \subset N} y_S\, .\, \alpha$ and this is at least

$\tilde{w}(N) = w(N) + \alpha$ if $(\sum_{S \subset N} y_S - 1)\alpha \geqq w(N) - \sum_{S \subset N} y_S\, w(S)$. This means that there is to every cost matrix C a real number $\alpha(C)$ such that $\tilde{w}$ has a non-empty core if and only if $\alpha \geqq \alpha(C)$. To prove this, we need the fact that for non-trivial solutions of $\sum_{S \subset N} y_S\, e_S = e_N$ with $y_S \geqq 0$ for all $S \subset N$, the inequality $\sum_{S \subset N} y_S > 1$ holds.

Proof : Since $\sum_{S \subset N} y_S \geqq \sum_{S:1 \in S} y = 1$ we have $\sum_{S \subset N} y_S \geqq 1$. If $\sum_{S \subset N} y_S = 1$ then $y_S > 0$ implies $1 \in S$. The same holds if we replace 1 by i. $y_S > 0$ implies $N \subset S$ and $\{y_S\}_{S \subset N}$ is the trivial solution.

3. In Potters et al. (1987) we gave an example of an asymmetric cost matric C satisfying the triangle inequalities $c_{ij} + c_{jk} \geqq c_{ik}$ for all $i,j,k \in N_0$ such that the associated traveling salesman game has nevertheless an empty core. Whether this can occur with symmetric cost matrices C too,is up to this moment an open question. But we know that only the triangle inequalities $c_{ij} + c_{jk} \geqq c_{ik}$ with $j = 0$ can be relevant, since the transition from C to $\tilde{C}$, as we investigated under (2), only influences these triangle inequalities. In fact, there is for each cost matrix C a real number $\beta(C)$ such that $\tilde{C}$ satisfies the triangle inequalities

$$\tilde{c}_{i0} + \tilde{c}_{0j} \geqq \tilde{c}_{ij} \quad \text{for all } i,j \in N$$

if and only if $\alpha \geqq \beta(C)$. The question above can be paraphrased as follows : Do we have $\alpha(C) \leqq \beta(C)$ for all *symmetric* cost matrices C ?

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CHAPTER XII

NONSYMMETRIC NASH BARGAINING SOLUTIONS

by Hans Peters

1. INTRODUCTION

Without any doubt the Nash bargaining solution (Nash(1950)) is the most well-known and popular solution concept in bargaining - in the theoretical literature as well as in applied and empirical work. Moreover the *family* of nonsymmetric Nash bargaining solutions has been the subject of much theoretical and applied work, much more so than nonsymmetric extensions of other solution concepts. Which could be the causes of or reasons for this popularity ? Empirical evidence for the Nash bargaining solution(s) certainly is not overwhelming and besides, lack of empirical results concerning other solution concepts makes any comparison difficult if not impossible. (For some recent empirical work see Svejnar (1986), or Van Cayseele (1987)). Further, quite some experiments have been conducted (see Roth and Malouf (1979) for an overview), but also these are not unambiguously conclusive in favor of the Nash solution(s). Even, earlier experiments by Crott (1971) point into the direction of the next popular solution, the Raiffa-Kalai-Smorodinsky solution (Raiffa(1953), Kalai and Smorodinsky (1975)). A psychological advantage of the latter solution might be its easy visualization: perhaps, it is easier for persons in an experiment to draw the straight line occurring in the definition of the Raiffa-Kalai-Smorodinsky solution than to draw the hyperbolas corresponding to the Nash product maximization. In actual computations, however, the Nash product maximization sometimes gives "nicer" formulas than the calculation of the Raiffa-Kalai-Smorodinsky solution (see, e.g., McDonald and Solow (1981)).

Thus, apart from historical or chronological considerations, the cause of the popularity of the Nash solution(s) must be a theoretical one. Indeed, there exist many axiomatic characterizations of the Nash solution(s), and there are also some noncooperative approaches to the bargaining problem which give support to the Nash solution(s). Further, Nash solutions have their twins in Cobb-Douglas utility or production functions. And the independence of irrelevant condition used by Nash to characterize his solution, also occurs in decision theory (e.g. Luce (1979)) or consumer theory (e.g. Weddepohl (1970), Peters and Wakker (1987)), and is related to the condition with the same name in social choice theory (Arrow (1951)).

The obvious conclusion is that there must be something special about the Nash solution(s). This chapter tries to support this conclusion by making explicit some of the items listed in the preceding paragraph. That is, we will describe quite a few axiomatic and other models for the Nash solution(s).

Briefly, the chapter is organized as follows. The first four of the remaining sections are mainly concerned with axiomatic considerations : in section 2 the independence of irrelevant alternatives condition is central, sections 3 and 4 consider other conditions and characterizations, and section 5 discusses properties with respect to changes in the status quo point as far as these relate to Nash solutions. In sections 6 and 7, some non-cooperative and economic models are described in which Nash solutions play an important role. Section 8 concludes.

This chapter contains almost no proofs : the reader will be referred to the original works. As a consequence we will not always formulate the strongest possible results since this would require additional proofs. For instance, the extreme solutions - the dictator solutions - are sometimes excluded. However, we do not think that this will influence any results in this chapter in an essential way.

2. PRELIMINARIES. INDEPENDENCE OF IRRELEVANT ALTERNATIVES

A (*2-person bargaining*) *game* is a pair (S,d) where d is an interior point of S and S is a closed convex subset of $\mathbb{R}^2$ such that $\max\{x_i : x \in S\}$ exists for i = 1,2, and such that $y \in S$ whenever $y \leqq x$ and $x \in S$. ($y \leqq x$ and $x \geqq y$ mean $y_i \leqq x_i$ for i = 1,2, $y < x$ and $x > y$ mean $y_i < x_i$ for i = 1,2.) In interpreting a game (S,d) we have in mind two players who either agree on an $x \in S$ giving utility x_i to player i, or fail to agree in which case they end up with utilities d_i. The point d is called *status quo point* or *disagreement point*. Closedness of S is required only for mathematical convenience (and is often implicitly satisfied, e.g. if an underlying set of "physical" alternatives is finite), convexity of S may come from the use of von Neumann-Morgenstern utility functions defined on lotteries between underlying alternatives, or e.g. from the use of concave utility functions in division problems. The last requirement in the definition of a game (S,d), called *comprehensiveness*, can be interpreted as *free disposibility of utility* for both players. Most of the results in this chapter, however, can be adapted to the case of no free disposibility without essential changes.

The definition of an n-*person bargaining game* is similar to the definition of a 2-person game : replace 2 by n everywhere. We will concern ourselves mostly with 2-person games. These have received much more attention in literature; reasons for this are the fact that many interesting conflict situations involve only two parties, and that in conflict situations with more than two parties the formation of coalitions, which are not allowed in n-person ("pure") bargaining games ("unanimity games"), often is a natural occurrence. Nevertheless, many results for 2-person games can be extended straightforwardly to n-person games; see further our remark in the final section.

Nash (1950) proposed to solve the bargaining problem by looking at the whole family of bargaining games. In general, an (n-*person bargaining*) *solution* is a map $\varphi : B^n \to \mathbb{R}^n$ with $\varphi(S,d) \in S$ for **every** $(S,d) \in B^n$ (*feasibility*) where B^n denotes the family of all n-person bargaining games. The *axiomatic approach* to bargaining consists of specifying a list of properties and trying to find the solution(s) satisfying these properties. Most theorists agree that this approach is best understood if one has an arbitrator in mind who considers these properties as reasonable and advises (or prescribes) the corresponding solution(s) to the players.

Before we can give the definition of the central concept of this chapter, namely the Nash solution and its nonsymmetric extensions, we need some additional notation. We write B instead of B^2. For $(S,d) \in B^n$, we denote by P(S) the *Pareto optimal subset of* S :

$P(S) := \{x \in S : \text{for all } y \in S, \text{ if } y \geqq x \text{ then } y = x\}$,

and by $P(S,d) := \{x \in P(S) : x \geqq d\}$ the *individually rational Pareto optimal* *set of* (S,d). Further $S_d := \{x \in S : x \geqq d\}$, and $W(S) := \partial S$, the boundary or *weakly Pareto optimal subset* of S.

DEFINITION 2.1. For every $(S,d) \in B$ and $0 < t < 1$, let $N^t(S,d)$ maximize the product $(x_1-d_1)^t(x_2-d_2)^{1-t}$ over S_d. We call $N^t : B \to \mathbb{R}^2$ a *(nonsymmetric) Nash solution*. We call $N := N^{\frac{1}{2}}$ *the Nash solution*. The *dictatorial solutions* $D^1, D^2 : B \to \mathbb{R}^2$ assign to each $(S,d) \in B$ the lower and upper point of $P(S,d)$, respectively. We also write $N^1 := D^1$ and $N^0 := D^2$. Further, $\mathcal{N} := \{N^t : 0 \leqq t \leqq 1\}$ and $\mathcal{N}^0 := \{N^t : 0 < t < 1\}$.

The Nash solution N was first proposed by Nash (1950), and the nonsymmetric Nash solutions N^t were (may be not first) proposed by Harsanyi and Selten (1972). Nash (1950, 1953) proposed the following properties for a solution φ on B.

WPO (*Weak Pareto optimality*) : $\varphi(S,d) \in W(S)$ for every $(S,d) \in B$.

IAUT (*Independence of positive affine utility transformations*) :
For all $a,b \in \mathbb{R}^2$ with $a > 0$ and every $(S,d) \in B$, we have $\varphi(aS + b, ad + b) = a\varphi(S,d) + b$. Here $ax := (a_1x_1, a_2x_2)$ for $x \in \mathbb{R}^2$, and $aT := \{ax : x \in T\}$ for $T \subset \mathbb{R}^2$.

SYM (*Symmetry*) : If $(S,d) \in B$ is *symmetric*, i.e. $d_1 = d_2$ and $S = \{(x_2,x_1) : x \in S\}$, then $\varphi_1(S,d) = \varphi_2(S,d)$.

IIA (*Independence of Irrelevant Alternatives*) : For all $(S,d), (T,d) \in B$ with $S \subset T$ and $\varphi(T,d) \in S$, we have $\varphi(T,d) = \varphi(S,d)$.

The WPO-property needs no further comment. The IAUT-property is reasonable and even compelling if the utility functions are of the von Neumann-Morgenstern type, so unique only up to positive affine transformations. As a matter of fact, the IAUT-property requires that the solution depend only on the players' preferences over the underlying "physical" alternatives and not on the particular representation of these preferences. The symmetry property reflects according to some authors equal bargaining ability, and according to others lack of such information as to enable us to distinguish between the players. Both arguments become questionable if knowledge of the underlying bargaining situation makes it possible to distinguish between the players. The well-known argument (e.g. Harsanyi (1977)) that all information must be incorporated into the players' utility functions is perhaps theoretically appealing, but tends to limit the applicability of the theory. Kalai (1977) presents a model which accounts for non-symmetric Nash solutions, see section 4 of this chapter. See also sections 6 and 7.

The IIA-property is the most discussed and criticized property in literature (e.g. Luce and Raiffa (1957), Kalai and Smorodinsky (1975)). For this moment, we only want to stress that in order to have a meaningful interpretation of IIA it should be taken literally : *ceteris paribus* (e.g. the players and their utility functions), if the set of underlying "physical" alternatives shrinks while *some* (there may be more) original solution alternative is still available, then the new solution alternative should be such an originally available solution alternative. The formulation of IIA reflects an implicit axiom present in many axiomatic models of bargaining and called the *Bargaining Theory Axiom* by Roemer (1986) : this axiom requires that the solution depend only on the bargaining game (i.e. the pair of objects in utility space) and not on the underlying "physical" bargaining situation. For instance, the IIA-property is assumed to hold also if the physical bargaining problem giving rise to (S,d) cannot be obtained by merely shrinking the set of alternatives in the problem underlying (T,d). A user of bargaining theory should be aware of this.

Nash (1950) proved the following theorem.

THEOREM 2.2. The Nash-solution $N : B \rightarrow \mathbb{R}^2$ is the unique solution with the properties WPO, IAUT, SYM, and IIA.

Three other properties for a solution φ on B, which need no further comments, are defined as follows.

(S)IR (*(Strong) individual rationality*) : $\varphi(S,d) \geqq (>)\ d$ for every $(S,d) \in B$.
PO (*Pareto optimality*) : $\varphi(S,d) \in P(S)$ for every $(S,d) \in B$.

The following two theorems for 2-person solutions are proved in Roth (1979) and de Koster et al. (1983), respectively.

THEOREM 2.3. N^0 is the family of all solutions with the properties SIR, IAUT, and IIA.

THEOREM 2.4. N is the family of all solutions with the properties IR, PO, IAUT, and IIA.

Theorem 2.4 is the first characterization of the family N, consisting of all the nonsymmetric Nash solutions and both dictator solutions, in this chapter : a large part of the remainder will present variations on this theme. The dictator solutions are the extremes in this family; they have many appealing properties, and it is only a pity that they are dictatorial. There is an interesting parallel here with Arrow's famous impossibility result (Arrow (1951)).

There exist many variations on the bargaining model and Theorem 2.4 that involve IIA or properties in the same spirit. Peters et al. (1983) consider *multisolutions* , which assign a subset of feasible outcomes to each bargaining game. Kaneko (1980) also considers multisolutions, and obtains a characterization of the "symmetric Nash correspondence" for *non-convex* bargaining games. Peters and Tijs (1983) consider so-called *probabilistic solutions* with an IIA-property; probabilistic solutions assign probabilities to the feasible points in a bargaining game. Patrone et al. (1987) show that in Theorem 2.4 the IIA-property may be considerably weakened if a weak continuity property (*Pareto continuity*) is added. This weaker version of IIA requires that if a game $(T,d) \in B$ is reduced by omitting all points above a certain utility level for player 2 and all points to the right of a certain utility level for player 1, then the solution point does not change provided it is still feasible. This variation or other ones on Theorem 2.4 may be useful in case the bargaining games in question are derived from "physical" situations in which the rather strong premises of the IIA-property are void of meaning.

A bargaining game can be viewed as a *decision problem* in which the decision maker consists of the two bargainers as a group, and in which the decision or compromise is the point assigned by some solution φ. In this context one might expect the "decision maker" to maximize certain "preferences"; formally, we say that the binary relation $\succcurlyeq$ on $\mathbb{R}^2$ *represents* φ if for every game (S,d) there is a unique point z with $z \succcurlyeq x$ for all x in S, and $z = \varphi(S,d)$. The following result is proved in Peters and Wakker (1987):

THEOREM 2.5. There exists a binary solution $\succcurlyeq$ on $\mathbb{R}^2$ representing φ if and only if φ satisfies IIA.

Theorem 2.5 holds for n-person solutions as well; its proof depends only on the intersection-closedness of the domain. Peters and Wakker (1987) also give a characterization of the family N^0 of nonsymmetric Nash solutions by characterizing the Nash product functions defining them. Like Theorem 2.5, this characterization emphasizes the "social welfare" character of the nonsymmetric Nash solutions; alternatively, the result can also be considered as a characterization of the Cobb-Douglas production or utility functions, for the case of convex compact production or consumption sets.

3. ALTERNATIVES FOR IIA

Our first aim in this section is to show that in Theorem 2.4 the independence of irrelevant alternatives property may be replaced by a property called independence of irrelevant expansions and introduced by Thomson (1981). We first give the definition. Here, φ is a two-person solution.

IIE (*Independence of irrelevant expansions*) : For every $(S,d) \in B$ there exists a vector $p \in \mathbb{R}^2_+$ with $p_1 + p_2 = 1$ such that :

(i) $p.(x-d) = p.(\varphi(S,d)-d)$ is the equation of a supporting line of S at $\varphi(S,d)$,

(ii) for all $(T,d) \in B$ with $S \subset T$ and $p.(x-d) \leqq p.(x-\varphi(S,d))$ for all $x \in T$, we have $\varphi(T,d) \geqq \varphi(S,d)$.

(x.y denotes the inner product $x_1y_1 + x_2y_2$, for $x,y \in \mathbb{R}^2$.)

Contrary to IIA, IIE says something about the way in which a game may be *expanded* without essentially changing the solution. A nonsymmetric Nash solution N^t $(0 < t < 1)$ satisfies IIE with equality in (ii), i.e. $\varphi(S,d) = \varphi(T,d)$; p is the normal vector of the supporting line separating a game from the hyperbola which is the level set of the maximal nonsymmetric Nash product. Only for the dictator-solutions in $\mathcal{N}$, it may actually happen that $\varphi(T,d) \neq \varphi(S,d)$ in (ii), namely if $p = (1,0)$ or $p = (0,1)$.

Stressing the social welfare aspect of a nonsymmetric Nash solution (see also the last part of the preceding section), we obtain an interpretation of IIE : the addition of feasible points with - apparently - lower social welfare does not influence the location of the maximal social welfare. This interpretation is related to the interpretation Shapley (1969) offers of the symmetric Nash solution $N = N^{\frac{1}{2}}$ as a compromise between "maximization of social welfare" and "sharing of social profit". Shapley's argument is based on the fact that the Nash solution point of a game is the unique Pareto optimal point where there is a supporting line ("maximization of social welfare") the slope of which is the negative of the slope of the straight line through that point and the status quo point ("sharing of social profit"). A proof of the following result can be found in Peters (1986b).

THEOREM 3.1. *$\mathcal{N}$ is the family of all solutions with the properties IR, PO, IAUT, and IIE.*

Theorem 3.1 concludes the first part of this section. Our third characterization of the family $\mathcal{N}$ will involve a weaker version of the following property for a solution φ on B.

SA (*Super-Additivity*) : For all (S,d) and $(T,e) \in B$ we have

$$\varphi(S + T,\ d + e) \geqq \varphi(S,d) + \varphi(T,e).$$

(Here $S + T := \{s + t : s \in S,\ t \in T\}$.)

Perles and Maschler (1981) give the following interpretation of the Super-Additivity property.

OBSERVATION. Suppose φ satisfies SA and IAUT. For any game consisting of a lottery on two games (S,d) and (T,e) in B, players who obey φ will prefer to reach an agreement before the outcome of the lottery is available. (*Proof*. Let $(p,1-p)$ be the distribution of the lottery, w.l.o.g. $0 < p < 1$.

If the players reach an agreement immediately, it must be $\varphi(V,f)$ where $(V,f) = p(S,d) + (1-p)(T,e)$. By IAUT and SA we obtain $\varphi(V,f) \geqq p\varphi(S,d) + (1-p)\varphi(T,e)$. The right hand side of this inequality is the expectation of the players from a delayed agreement.)

Peters (1986a) offers another interpretation of the Super-Additivity property. Suppose the players have to play two bargaining games (on two different issues, say). Intuitively, it may be advantageous for both players to play the games simultaneously rather than separately because then agreements like "if you grant my wishes in one game then I will grant yours in the other game" can be effective. If the simultaneous game can be expressed as the sum of the separate games then this intuition is equivalent to requiring a solution to be super-additive; necessary and sufficient conditions for this "sum-property" are given in Peters (1985). Perles and Maschler (1981) characterize their so-called *Super-Additive* solution on the class of games where every Pareto-optimal outcome is individually rational; on a larger class (e.g. B) they obtain an impossibility result. For the interpretation of Peters (1986a) it is essential to consider all games since concessions made by one player to his opponent in one game can be, typically, non-individually rational for the player who makes them. Peters (1986a) proposes a few weakenings of Super-Additivity, of which the following one is relevant in this chapter. Here, we call an $(S,d) \in B$ *smooth at* $x \in S$ if S has a unique supporting line at x.

RA (*Restricted Additivity*) : For all (S,d) and (T,e) in B such that S and T are smooth at $\varphi(S,d)$ and $\varphi(T,e)$ respectively, and $\varphi(S,d) + \varphi(T,e) \in P(S + T)$, we have $\varphi(S + T, d + e) = \varphi(S,d) + \varphi(T,e)$.

All members of $\mathcal{N}$ satisfy RA; RA is strictly weaker than SA since for instance the Nash solution does not satisfy SA, and SA implies RA. Theorem 3.2 below gives a third characterization of the family $\mathcal{N}$. First we need to define another property for a solution φ; this property was mentioned before, in the preceding section.

PCO (*Pareto continuity*). Let (S,d), (S^1,d^1), (S^2,d^2),... $\in B$ be a sequence of games with $d^n \to d$ and $\mathrm{conv}(P(S^n)) \to \mathrm{conv}(P(S))$ (w.r.t. the Hausdorff metric, see Peters (1986b, section 19)). Then $\varphi(S^n,d^n) \to \varphi(S,d)$.

Pareto continuity is weaker than *continuity* : in the latter property instead of $\mathrm{conv}(P(S^n)) \to \mathrm{conv}(P(S))$ only $S^n \to S$ is required. PCO is strictly weaker than continuity : the dictator solutions are Pareto continuous but not continuous. We can now state the announced theorem, a proof of which can be found in Peters (1986a).

THEOREM 3.2. N is the family of all solutions with the properties IR, PO, IAUT, PCO, and RA.

We conclude this section with a fourth axiomatic characterization of the family N. We will call the main property involved the *multiplication property*. We have chosen this mathematically sounding name since we will restrict ourselves in this case to formulating the property without giving an interpretation. Interpretations can be found in Binmore (1982) and Peters (1986b) : both interpretations, we think, are not quite convinving.

For notational simplicity, we restrict ourselves to a class B^0 of objects which we also call bargaining games. An $S \subset \mathbb{R}^2$ is an element of B^0 if $S = T \cap \mathbb{R}^2_+$ for some $(T,0) \in B$: so the status quo point is fixed at 0 and all non-individually rational points are discarded. Most definitions can be modified for B^0 in a straightforward manner. For $S,T \in B^0$ we denote $ST := \{st : s \in S, t \in T\}$.

MP (*Multiplication property*) : For all $S,T \in B^0$, if $ST \in B^0$, then $\varphi(ST) = \varphi(S)\varphi(T)$.

Note that the condition $ST \in B^0$ cannot be omitted since ST does not have to be convex.

THEOREM 3.3. N is the family of all solutions on B^0 with the properties PO and MP.

In theorem 3.3 $\mathcal{N}$ is characterized by only two properties, but notice that IR is implied by our restriction to B^0 and IAUT follows easily from MP. The result can be formulated for solutions on B at the price of a few extra notations; for this and for a proof, see Peters (1986b).

So far we have formulated four axiomatic characterizations of Nash solutions. In the next section we discuss some further axiomatic and nonaxiomatic odds and ends of solutions in $\mathcal{N}$.

4. REPLICATIONS AND RECURSIVITY

In this section we first discuss a model proposed by Kalai (1977), who derives nonsymmetric Nash solutions by considering certain *replications* of two-person bargaining games. Then we consider another axiomatic characterization of the symmetric Nash solution obtained by Van Damme (1986), in which a property called *recursivity* is used.

Replication of two-person bargaining

Let $(S,d) \in B$ and let m and ℓ be two positive integers. The (m,ℓ)-*replication of* (S,d) is defined as $(S',d') \in B^{m+\ell}$ where S' is the comprehensive hull of $\{y \in \mathbb{R}^{m+\ell}$: there is an $x \in S$ such that $y_i = x_1$ for $1 \leq i \leq m$ and $y_i = x_2$ for $m+1 \leq i \leq m+\ell\}$, and $d'_i = d_1$ if $1 \leq i \leq m$, $d'_i = d_2$ if $m+1 \leq i \leq m+\ell$. (The *comprehensive hull of* $T \subset \mathbb{R}^n$ is defined as $T - \mathbb{R}^n_+$.) The n-*person symmetric Nash solution* is defined analogous to the (2-person) Nash solution; i.e., for every $(S,d) \in B^n$ it picks out the unique point where the product $\Pi_{i=1}^n (x_i-d_i)^\alpha$, with $\alpha = \frac{1}{n}$, is maximized on $P(S,d)$. Note that, for a game $(S,d) \in B$ with (m,ℓ)-replication $(S',d') \in B^{m+\ell}$, the $(m+\ell)$-person symmetric Nash solution point z of (S',d') has the first m coordinates equal to the first coordinate of $N^t(S,d)$ and the last ℓ coordinates equal to the second coordinate of $N^t(S,d)$, where $t := \frac{m}{m+\ell}$; this follows by writing down the Nash products which have to be maximized. Any $0 \leq t \leq 1$ can be (at least) approximated by a quotient $\frac{m}{m+\ell}$ with m and ℓ positive integers. This is, briefly, the main insight of Kalai (1977) : Every member of $\mathcal{N}$ can be (at least approximately) derived from the symmetric Nash solution applied to replications of two-person bargaining games with the appropriate dimensions. (Kalai does not include the dictator-solutions but there is no problem in including these as well.)

The meaning of the idea of replicating games in this way is well illustrated by Kalai's own example which we quote here.

EXAMPLE. Consider two players, 1 and 2, who have one dollar to divide between them. If they do not come to an agreement on how to divide the dollar they lose it, and each receives nothing. Each player has utility y for y units of money that he receives. Nash's solution states that the dollar should be divided evenly between the players. Now assume that player 1 has an enthusiastic supporter 1', (say a mother), who also has utility y for y units of money that 1 receives. In addition 1' has to agree to the decisions that 1 makes. The 2-person game becomes a 3-person game which is a (2,1)-replication of the original game. If the symmetric Nash solution is considered as reasonable, then we have $(\frac{2}{3},\frac{2}{3},\frac{1}{3})$ as solution point in this 3-person game; this corresponds to giving $\frac{2}{3}$ of the dollar to player 1, so to the point assigned by the solution N^t with $t = \frac{2}{3}$ in the original 2-person game. (We may add to Kalai's example that a more plausible interpretation for this outcome would be that player 1 and his mother split the $\frac{2}{3}$ dollar evenly.)

Recursivity

In section 2 (second paragraph before Theorem 2.5) we have indicated an axiomatic characterization of the family N in which the IIA-property was weakened at the price of adding a weak continuity property. Van Damme (1986) does a similar thing by weakening IIA to a property called recursivity while "adding" the following property for a solution φ.

RS (*Risk Sensitivity*) : For all (S,d) and (S',d') in B such that (S',d') arises from (S,d) by applying a nondecreasing concave function $k : \mathbb{R} \to \mathbb{R}$ to the i-th coordinates of points in S (i = 1 or 2) with $k(d_i) = d'_i$, we have $\varphi_j(S,d) \leqq \varphi_j(S',d')$ $(j \neq i)$.

This property states that replacing a player (called player i here) by a *more risk averse* player (according to the Arrow-Pratt measure of risk aversion) is not disadvantageous for that player's opponent (j here). The property was first proposed in Kihlstrom et al. (1981), who also showed that the symmetric Nash solution is risk sensitive. De Koster et al. (1983) showed that every member of $\mathcal{N}$ is risk sensitive.

Let F denote the family of all individually rational, Pareto optimal, risk sensitive, symmetric 2-person bargaining solutions. For $(S,d) \in B$ let $(S,d)_F := \{\varphi(S,d) : \varphi \in F\}$. Van Damme (1986, Proposition 1) shows that $(S,d)_F$ is a closed connected subset of $P(S,d)$. So $((S,d)^F,d) \in B$ where $(S,d)^F$ denotes the comprehensive hull of $(S,d)_F$. Van Damme (1986) considers the following property.

R (*Recursivity*) : For all $(S,d) \in B$, $\varphi(S,d) = \varphi((S,d)^F,d)$.

The recursivity property requires that the solution should not depend on outcomes which cannot be obtained as the result of any solution in a "reasonable" family, in this case F. The following result is Proposition 5 in Van Damme (1986).

THEOREM 4.1. The Nash solution N is the unique solution with the properties IR, PO, SYM, RS, and R.

If we add SYM in Theorem 2.4 and compare the result with Theorem 4.1, we notice that IIA has been replaced by R, and IAUT by RS. Indeed, for a solution in F, IIA implies R (as is straightforward to verify); further, Kihlstrom et al. (1981) have already shown that IAUT is implied by the combination of PO and RS. Thus, in Theorem 4.1, IIA is weakened to R at the price of the substitution of RS for IAUT. We conclude with the conjecture that Theorem 4.1 can be extended to the whole family $\mathcal{N}$ by modifying the symmetry condition, e.g. by requiring solutions to assign the same point to same "standard" game like (T,0) with T the convex comprehensive hull of (1,0) and (0,1).

5. THE STATUS QUO POINT

Up to now the premises of important properties like IIA, IIE, and R, had in common that in a game (S,d) the set of feasible outcomes S was changed (to a super- or subset) whereas the status quo point d remained fixed. More generally, none of the properties formulated so far focuses on possible changes in the solution outcome as a result of changing the status quo point. In Peters (1986c) characterizations are given of some two-person bargaining solutions with the aid of properties in which the status quo point plays a central role. This also leads to a characterization of the symmetric Nash solution. We need the next two properties for a two-person solution φ.

INIR (*Independence of Non-Individually Rational outcomes*) : For every game (S,d), $\varphi(S,d) = \varphi(T,d)$ where T is the comprehensive hull of the individually rational set S_d of (S,d).

LIN (*Linearity*) : For every game (S,d), if d' is an interior point of S on the straight line through d and $\varphi(S,d)$, then $\varphi(S,d') = \varphi(S,d)$.

The INIR-property is a very mild kind of "independence of irrelevant alternatives". Linearity can be interpreted as a kind of proportionality : as long as the proportion of the relative gains of the players (relative w.r.t. the status quo utilities) does not change while the status quo point changes, the solution outcome should not change. From Peters (1986c, Theorem 3.1) can be derived :

THEOREM 5.1. The Nash solution N is the unique 2-person bargaining solution with the properties PO, INIR, SYM, IAUT, and LIN.

In this theorem the combination of SYM and IAUT may be replaced by a property called *Split-the-difference* (see Peters (1986c)) : suppose $x,y \in \mathbb{R}^2$ with $x_1 < y_1$ and $x_2 > y_2$ and let T be the convex comprehensive hull of x and y, then the Split-the-difference property requires $\varphi(T,(x_1,y_2)) = \frac{1}{2}x + \frac{1}{2}y$. We further note that PO and INIR together imply individual rationality.

Peters and Van Damme (1987) show that in Theorem 5.1 the linearity property may be replaced by a weaker property (which requires d' in the formulation of LIN to lie in between d and $\varphi(S,d)$) if continuity of φ with respect to changes in the status quo point (for a fixed set of feasible outcomes) is added. Let us call the latter property : SC (*Status quo point continuity*). Instead of the former property, we formulate the following property :

MSC (*Mid Status quo point Convexity*) : For all (S,d), (S,d') $\in$ B with $\varphi(S,d) = \varphi(S,d')$ we have $\varphi(S,d) = \varphi(S,\frac{1}{2}d + \frac{1}{2}d')$.

MSC may be interpreted in the following way. Suppose that in a game the status quo point is still unknown and that the players consent to the same agreement regardless whether d or d' will be the status quo point; suppose they know that each of these two points has a probability of $\frac{1}{2}$ of becoming the status quo point; then the expected status quo point should induce the same agreement. From Peters and Van Damme (1987) the following result can be derived :

THEOREM 5.2. The Nash solution N is the unique 2-person bargaining solution with the properties PO, INIR, SYM, IAUT, SC, and MSC.

Finally, we conjecture again that also in these two theorems characterizations of the whole family $\mathcal{N}$ are obtained by omitting the symmetry property, or modifying the split-the-difference property mentioned after Theorem 5.1.

6. ECONOMIC MODELS FOR NASH SOLUTIONS

In recent years there have been quite some applications of Nash solutions to economic problems. We mention the work of McDonald and Solow (1981), Grout (1984), and Svejnar (1986), but there are many others. Often, these works are not particularly concerned with the question why the Nash solution or nonsymmetric Nash solutions should be appropriate for the economic model or problem under consideration.

In Roemer (1986), this author considers certain economic environments - distribution problems - and formulates axioms for distribution schemes - bargaining solutions - in terms of these economic environments. In this way he arrives at characterizations of some well-known solution concepts, among which is the Nash solution. In a technical sense, these axioms are weaker than the corresponding axioms (properties) in general bargaining theory as presented in this paper so far, but this is compensated by the richer economic structure of the underlying bargaining situation, i.c. division problem. The advantage of this approach over the works indicated in the preceding paragraph, is that the properties which may guide us to choose a particular solution concept are formulated in economic terms and therefore, presumably have clear economic contents. The availability of such economic information and its use in characterization theorems clearly is also an advantage of Roemer's approach over the general abstract model. In Roemer's words, the general model adopts the *Bargaining Theory axiom* which states that bargaining solutions do not distinguish between bargaining situations which have the same image in utility space, i.e. give existence to the same game (S,d). On the other hand, Roemer's characterization results are valid only for a(n admittedly broad) class of distribution problems. For other economic problems, the work has to be done anew.

Roemer's results stipulate the importance of justifying the use of a specific game-theoretic solution concept in terms of the specific economic problem to which it is applied. Thus, as an example, if the independence of irrelevant alternatives property does not have any sensible meaning in some economic model then one should hesitate using the Nash solution; anyway, even if one wants to use it, one should not appeal to Theorem 2.3 (or 2.4) for a justification.

Broadly speaking, bargaining game theory can be applied to economic models involving more parties where a choice has to be made from some set of feasible outcomes and where a reasonable interpretation can be given to an economic event serving as the status quo outcome. E.g., in an exchange economy it is generally agreed upon that a choice must be made from the contract curve; here *no trade* : the agents are left with their initial endowments, may serve as the status quo event. For such models other kinds of solution concepts exist; the most well-known is *competitive* (or

Walrasian) equilibrium. An interesting question is whether there exist relations between bargaining solutions in general, or particular solutions like Nash solutions, and a concept like competitive equilibrium. A partial affirmative answer to this question is given in Peters (1986b, section 13), who considers an economy in which two agents with different initial money endowments are bargaining over the division of a continuum of goods which they can buy together. A characterization of all competitive equilibria, and of all bargaining solutions giving rise to equilibrium allocations, is obtained. Special attention is paid to Nash solutions the weights of which turn out to correspond to the initial money endowments of the players.

A related model was already presented in Gale (1960). The remainder of this section is devoted to a discussion of that model.

In this model there are m *consumers* labelled $C_1, C_2, \ldots, C_m$ and n *goods* labelled $G_1, G_2, \ldots, G_n$. The utility of 1 unit of good G_j to consumer C_i is $\alpha_{ij} \geqq 0$, and we suppose that utility is additive : if $y = (y_1, y_2, \ldots, y_n) \geqq 0$ is a *bundle of goods*, then $\sum_{j=1}^{n} \alpha_{ij} y_j$ is the utility of y to consumer C_i. Each consumer C_i has an amount of money b_i, which he is willing to spend entirely on the n goods. There is one unit of each good available, and $\sum_{i=1}^{n} b_i$ is exactly enough to buy all the goods. We suppose further that each good is wanted (for every j there is an i with $\alpha_{ij} > 0$) and that each consumer wants something (for every i there is a j with $\alpha_{ij} > 0$). A *price vector* $p = (p_1, p_2, \ldots, p_n)$ is a nonnegative vector with $\sum_{j=1}^{n} p_j = \sum_{i=1}^{m} b_i$. A *competitive equilibrium* consists of a price vector $p = (p_1, p_2, \ldots, p_n) \geqq 0$ and a bundle of goods y^i for each consumer C_i such that

(i) y^i maximizes C_i's utility $\sum_{j=1}^{n} \alpha_{ij} y_j$ subject to the budget constraint $p.y = b_i$, for every i. (. denotes the inner product.)

(ii) $\sum_{i=1}^{m} y^i = (1, 1, \ldots, 1)$, that is, all goods are bought completely.

Let the price vector p and the bundles y^i form a competitive equilibrium. Suppose that the quantities y^i_ℓ and y^i_k for consumer i are positive but unequal to 1. Since y^i maximizes C_i's utility subject to the constraint $p.y = b_i$, we must have $y^i_\ell \, p_\ell^{-1} = y^i_k \, p_k^{-1}$ since otherwise C_i could improve upon his bundle without violating his budget constraint by buying more (less) of the good with the higher (lower) ratio of marginal utility over price. For the same reason, this ratio must not be higher for the goods of which he buys nothing, but it may be higher for a good of which he buys everything but which does not exhaust his budget.

EXAMPLE. (Cf. Gale (1960, p. 287).) Consider the following matrix for the (marginal) utilities α_{ij} and money endowments b_i :

	G_1	G_2	G_3	b_i
C_1	4	3	1	1
C_2	2	3	2	1
C_3	3	1	2	1

The vector $p = (\frac{6}{5},1,\frac{4}{5})$ and the bundles $y^1 = (\frac{5}{6},0,0)$, $y^2 = (0,1,0)$, $y^3 = (\frac{1}{6},0,1)$ form a competitive equilibrium. For C_1, the marginal utility / price ratios are $3\frac{1}{3}$, 3, and $1\frac{1}{4}$ for G_1, G_2, and G_3, respectively; for C_2 and C_3 these numbers are $1\frac{2}{3}$, 3, and $2\frac{1}{2}$, and $2\frac{1}{2}$, 1, and $2\frac{1}{2}$, respectively. These numbers are in accordance with our remarks made above. Note that also the vector $\hat{p} = (\frac{13}{10},1,\frac{7}{10})$ and the bundles $\hat{y}^1 = (\frac{10}{13},0,0)$, $\hat{y}^2 = (0,1,0)$, and $\hat{y}^3 = (\frac{3}{13},0,1)$ constitute a competitive equilibrium. Again the marginal utility / price ratios are consistent with our earlier remarks; note in particular that for consumer C_3 these ratios are $2\frac{6}{7}$ for G_3 and $2\frac{4}{13}$ for G_1. Still, C_3 buys also a quantity of G_1 since he has already bought all available G_3 and has some money left.

In this example, considering the competitive equilibrium $(\hat{p},\hat{y}^1,\hat{y}^2,\hat{y}^3)$, we could as well omit the third good G_3. We are then left with a 3 consumers - 2 goods model where the consumers C_1, C_2 and C_3 have money endowments 1,1, and $\frac{3}{10}$, respectively, and where $\hat{p} = (\frac{13}{10},1)$, $\hat{y}^1 = (\frac{10}{13},0)$, $\hat{y}^2 = (0,1)$, and $\hat{y}^3 = (\frac{3}{13},0)$ still constitute a competitive equilibrium. For this competitive equilibrium the general idea that a consumer buys only those goods for which the marginal utility / price ratio is maximal, is restored. In the same way, any competitive equilibrium can be reduced in this way to what we shall call : an *essential competitive equilibrium*, and which consists of a price vector $p = (p_1,p_2,\ldots,p_n)$ and a bundle of goods y^i for each consumer C such that, besides (i) and (ii) in the definition of a competitive equilibrium, we moreover have :

(iii) $p > 0$

(iv) For every $j = 1,2,\dots,n$ and every $i = 1,2,\dots,m$, if $y_j^i > 0$ then $\alpha_{ij}p_j^{-1} = \max_k\{\alpha_{ik}p_k^{-1}\}$.

In order to justify requirement (iii), suppose that for a competitive equilibrium $p_j = 0$ for some j. Then there can be only one consumer who wants G_j, i.e. $\alpha_{ij} > 0$ for only one C_i, and that consumer will receive all of G_j. But then we could as well omit G_j from consideration, i.e. give it all to C_i for nothing, without essentially changing the competitive equilibrium. We will restrict further attention to essential competitive equilibria and describe Gale's (1960, p.281 ff.) results.

We normalize money and prices so that $\sum_{i=1}^m b_i = \sum_{j=1}^n p_i = 1$ and consider the optimization problem

$$\text{Maximize } \Pi_{i=1}^m \left(\sum_{j=1}^n \alpha_{ij}y_j^i\right)^{b_i}$$

$$\text{s.t. } \sum_{j=1}^n y_j^i = 1 \text{ for every } i = 1,2,\dots,m.$$

For each collection bundles $y^1,\dots,y^m$ satisfying the constraints of this problem we can calculate the point with i-th coordinate $\sum_{j=1}^n \alpha_{ij}y_j^i$. Taking the comprehensive hull of all such points, we obtain an m-dimensional bargaining game, where we take the origin as status quo point. So solving the above problem is the same as calculating the m-*person nonsymmetric Nash bargaining solution* with weights $b_1,b_2,\dots,b_m$, for the corresponding bargaining game. (The definition of this bargaining solution is obvious.) Let $\bar{y}^1, \bar{y}^2,\dots, \bar{y}^m$ be a maximizing combination of bundles of goods : existence is no problem since a continuous function is maximized on a compact set. For every $j = 1,2,\dots,n$ let $p_j = \max \alpha_{ij}b_i(\sum_{j=1}^n\alpha_{ij}\bar{y}_j^i)^{-1}$. Then Gale (1960, Theorem 8.5) shows that $p = (p_1,p_2,\dots,p_n)$ together with the bundles $\bar{y}^i$ constitute an essential competitive equilibrium ! This result is of special importance because, as Gale also shows (Theorem 8.6), prices in an essential competitive equilibrium are *unique*.

We conclude that this model provides a "noncooperative" foundation for the use of nonsymmetric Nash bargaining solutions. This foundation is especially attractive since the weights b_i which are usually - and rather vaguely - interpreted as "bargaining ability" measures, get a clear meaning: they are the initial money endowments of the agents. In more general terms, the model gives a relation between a noncooperative solution concept (competitive equilibrium) and a cooperative concept (bargaining solution).

Gale's treatment of this model contains a flaw in that he considers (what we have called) essential competitive equilibria without making this assumption explicit. Further, the assumptions on the nature of the consumers' utility functions are rather restrictive. It is not difficult to derive similar results for the case where the utility functions are homogeneous of degree at most one, and to exclude nonessential competitive equilibria by appropriate extra conditions. The interested reader may verify this remark.

The next section will be concerned with other, strategic, noncooperative models for Nash solutions.

7. STRATEGIC MODELS FOR NASH SOLUTIONS

Nash invented two fundamental concepts in game theory. One of these - the Nash bargaining solution - is the subject of this chapter, and the other one - the *Nash equilibrium* concept which is at least as important - is the subject of about half of the chapters of this book. The Nash bargaining solution was proposed by Nash in his 1950 paper, and the equilibrium concept in his 1951 paper. Nash (1953) is about the Nash bargaining solution, but the author is also concerned with trying to give a noncooperative foundation to his solution, i.e. with describing a noncooperative game corresponding to a given bargaining game such that the Nash bargaining solution outcome coincides with "some" equilibrium of the noncooperative game. There is general agreement in literature that he did not fully succeed; still, also in this respect he has led the way for later work by other authors. For more details on Nash's approach, we refer to Sutton (1986).

Thus, Nash himself was the first one to contribute to - what is called by some authors - the *Nash program*. In a broad sense, this program looks for noncooperative, strategic implementations of cooperative (normative) concepts in game theory. In a narrow sense, it means the same thing but then for the Nash bargaining solution(s). The economic model of Gale described in the previous section of this chapter is an example. There, one can imagine an arbitrator defending his forcing a nonsymmetric Nash solution on the consumers' distribution problem by pointing out that it leads to a competitive equilibrium; moreover, if the arbitrator is confident that the bargaining of the consumers will lead to a competitive equilibrium automatically, then he does not need to force the nonsymmetric Nash solution in question upon the consumers.

Harsanyi (1956) recognized the relation that exists between a bargaining process discussed by Zeuthen (1930) and the Nash bargaining solution. In the *Nash demand game* corresponding to a two-person bargaining game (S,d), each player proposes an outcome in P(S,d). If the proposals are incompatible, so if the outcome by player 1 is lower on P(S) than the outcome proposed by player 2, then in the Harsanyi-Zeuthen model the player with the lower *risk limit* will make a concession, that is propose a new outcome which is better for his opponent. The risk limit is the net loss of conceding to the proposal of one's opponent divided by the net loss in case disagreement occurs and d is the final outcome. It is easy to show that this model leads to the symmetric Nash bargaining solution provided there is some lower bound to the concessions. Harsanyi also proposed a two-step version of this model in which intermediate concessions are excluded, and of which the unique *Nash equilibrium* is equal to the pair of *maximum strategies* in which each player proposes the Nash bargaining solution outcome. Peters (1986b, section 14) provides a model which implements nonsymmetric Nash bargaining solutions by a mixture of the ideas present in the Harsanyi-Zeuthen model and the Kalai replication model (see section 4 of this chapter). There have been some variations and many criticism on the Harsanyi-Zeuthen model in literature. Apart from the mentioned works, we refer to Harsanyi (1977), Roth (1979), Battinelli et al. (1986).

In Anbar and Kalai (1978), again the Nash demand game is played. Now each player thinks that his opponent proposes randomly according to a uniform distribution. For each player, maximizing his expected utility means demanding the Nash bargaining solution outcome. For a game (S,d) let $\bar{x}$ be the boundary point of S with second coordinate d_2 and let $\bar{y}$ be the boundary point of S with first coordinate d_1. Suppose player 1 proposes an outcome x with first coordinate $x_1 \in [d_1,\bar{x}_1]$ and thinks that the second coordinate of player 2's proposal is drawn from a uniform distribution over $[d_2,\bar{y}_2]$. Then player 1's expected utility equals

$$x_1 \frac{x_2-d_2}{\bar{y}_2-d_2} + d_1 \frac{\bar{y}_2-x_2}{\bar{y}_2-d_2} = d_1 + \frac{(x_1-d_1)(x_2-d_2)}{\bar{y}_2-d_2}$$

; here, a "one shot" bargaining model is assumed. So player 1 maximizes his expected utility by proposing the Nash bargaining solution outcome, and so does player 2. In this model, the bargaining game is reduced to two single-player decision problems. For further discussion and criticisms on this model, see Roth (1979, p. 25 ff.)

Another model was proposed by Van Damme (1986). In this model, the players start with a 2-person bargaining game $(S^0,d) = (S,d)$ at time 0. The game is played in successive rounds and lasts at least 1 round. If the game is not over at time t (t = 0,1,2,...), both players must choose an outcome in the remaining game (S^t,d) attainable by a solution belonging to a specific class of "well-behaved" solutions; a solution is "well-behaved" if it is Pareto optimal, symmetric, independent of positive affine utility transformations, and risk sensitive (; see sections 2 and 4 for the definitions of these properties). If the chosen outcomes are incompatible in (S^t,d) which is the case if the outcome chosen by player 1 lies below the outcome chosen by player 2, then the players continue with the game (S^{t+1},d) where $S^{t+1} \subset S^t$ arises by deleting all points of S^t which are either above player 2's choice or to the right of player 1's choice : apparently - and this is the justification for the prescribed bargaining procedure - these outcomes were already given up by either player 2 or player 1. If a player always adopt the Nash bargaining solution outcome then obviously (in view of IIA) he will never have to "concede". Van Damme's surprising result is that this is the only outcome with this property, so no player can do better than always propose the Nash bargaining solution outcome. Actually, Theorem 4.1 in this chapter is an immediate consequence of this result, as Van Damme shows ((1986, Proposition 5)). Van Damme's model makes the IIA-property or rather a weak version of it - in fact, it is reminiscent of the weaker version of IIA mentioned in the discussion following Theorem 2.4 - operational in a bargaining procedure.

This section would not be complete without a discussion of Rubinstein's bargaining model (Rubinstein (1982)). Indeed, also that model leads to the Nash solution(s), as is pointed out in Binmore et al. (1986). We will now describe these results in more detail, thereby making a few assumptions on the players' preferences which do not substantially restrict the general ideas.

Two players have to agree on the division of one unit of a perfectly divisible good, say a cake. A player's preferences depend not only on the slice of the cake he manages to obtain, but also on the time. If player 1 gets a portion x between 0 and 1 of the cake at time t = 0,1,2,..., then his utility equals $\delta_1^t u(x)$ where $0 \leqq \delta \leqq 1$ is player 1's discounting factor (we define $0^0 := 1$) and $u : [0,1] \to \mathbb{R}$ is a continuous increasing concave function with u(0) = 0 and u(1) = 1. There are a number of hidden assumptions on player 1's preferences here, among which is stationarity in time (see Rubinstein (1982), Fishburn and Rubinstein (1982)). Similarly, player 2's utility for receiving x at time t equals $\delta_2^t v(x)$ where δ_2 and v have similar properties as δ_1 and u.

The game is played in successive rounds. At times t = 0,2,4,..., player 1 makes a proposal concerning the division of the cake and player 2 answers yes or no. At times t = 1,3,5,..., player 2 makes a proposal and player 1 answers yes or no. The game ends as soon as and only if some player says yes at some time. A *strategy* for a player is a complete plan which tells him what to do (i.e. which offer to make, or which answer to give) at each point of time (on to infinity) and for *each* history of offers and answers until that point of time. A pair of strategies constitute a *Nash equilibrium* if no player can gain from unilaterally deviating from his strategy. It turns out that for the model under consideration the Nash equilibrium is too weak a concept : *each* division x for player 1 and 1-x for player 2 can be obtained by a Nash equilibrium pair of strategies, as follows. Player 1 always proposes the division (x,1-x), and rejects any proposal which gives him less than x; player 2 always proposes (x,1-x) and rejects any proposal in which he gets less than 1-x. It is easy to verify that these two strategies form a Nash equilibrium; they result in the division (x,1-x) at time 0. We call the described pair of strategies a *simple* Nash equilibrium for (x,1-x).

A pair of strategies actually determines a "path" in the "game tree" :

although a strategy contains a prescription for every point in the tree, that is for every point of time and every history of offers and answers, most points in the tree are never reached. At each point of time and given a certain history the players can be said to play a new game ("subgame") from that time on; in the game tree, at each node a subgame begins. Even if the players play a Nash equilibrium pair of strategies, these strategies do not have to induce a Nash equilibrium in every subgame. If they do, however, then we call the pair of strategies a *perfect Nash equilibrium* (see Selten (1975)).

EXAMPLE. Take $u(x) = x$, $v(x) = x, \delta_2 = 0.9$. Suppose the players adopt the simple Nash equilibrium for (0.5,0.5). Suppose player 1 proposes the division (0.54,0.46) at time 0. (This leads to a node in the game tree off the equilibrium path : it may be reached for instance by accident, player 1 making a mistake.) If player 2 sticks to his equilibrium strategy, he rejects and obtains 0.5 in round 1 which means 0.9 . 0.5 = 0.45 in terms of utilities : this is worse than the 0.46 which he can get at time 0. So the simple Nash equilibrium for (0.5,0.5) is not a perfect equilibrium : we have described a subgame in which it is better for player 2 to deviate, even if player 1 plays his original strategy from time t = 1 on.

Rubinstein has shown that for the described game, if at least one discounting factor is positive and at least one is smaller than 1, there exists a unique division of the cake which can be obtained by a perfect Nash equilibrium. Consider the following system of equations :

$$u(y) = \delta_1 u(x), \quad v(1-x) = \delta_2 v(1-y).$$

Let (x^*,y^*) be the unique solution, then the pair of strategies in which player 1 (2) always proposes $(x^*,1-x^*)((y^*,1-y^*))$ and rejects any offer which gives him less then $y^*(1-x^*)$ is a perfect Nash equilibrium.

If $u(x) = v(x) = x$, then the equations above reduce to $y = \delta_1 x$ and $1-x = \delta_2(1-y)$. E.g., if player 1 is offered y, then he is indifferent between receiving y (saying yes) and receiving his own proposal only in the next round. A similar reasoning applies to player 2. Solving the equations for this case, we obtain

$$x^* = \frac{1-\delta_2}{1-\delta_1\delta_2}, \quad y^* = \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}.$$

The numbers $1-\delta_1$ and $1-\delta_2$ can be regarded as the *interest rates* of the players : If player i gets x tomorrow instead of today, that delay costs him $(1-\delta_i)x$. Suppose the time between successive rounds falls to 0. This means that the interest rates (measured over the time between bargaining rounds) decrease to zero but their proportion $\alpha := (1-\delta_1)(1-\delta_2)^{-1}$ remains constant. In our example with $u(x) = v(x) = x$, the numbers x^* and y^* both converge to $(1+\alpha)^{-1}$. So if the perfect equilibrium strategies described above are played, this results in the limit division $((1+\alpha)^{-1}, \alpha(1+\alpha)^{-1})$ if the time between successive bargaining rounds approaches 0. This limit division is also obtained by applying a nonsymmetric Nash solution N^t, with $t = (1+\alpha)^{-1}$, to the 2-person bargaining game with all possible divisions of the cake as Pareto optimal outcomes and 0 ("no one gets any cake") as status quo point. Moreover, Binmore et al. (1986) show that this fact is true for general u and v. Note further that $(1+\alpha)^{-1} = r_2(r_1+r_2)^{-1}$ where $r_i = 1-\delta_i$ is the interest rate of player i. So we have an interpretation of the weight of a nonsymmetric Nash solution in terms of interest rates.

We conclude that Rubinstein's model implements the nonsymmetric Nash solutions as limiting cases. As it were, in the limit all the bargaining is condensed in one moment of time; and this is, in fact, an often heard argument to defend the axiomatic approach. So here the axiomatic and the strategic approaches to bargaining are in striking harmony.

Finally, it is worthwile to note that the interest rate $r_i = 1-\delta_i$ can also be viewed as the probability assessed by player i that the bargaining process breaks down (and perpetual disagreement or status quo results) between two successive rounds. This gives a different interpretation to the model without essentially changing the results in a technical sense. Binmore et al. (1986) provide more details.

8. CONCLUSION

In this chapter we have reviewed many axiomatic, economic, strategic (noncooperative), and other models which in some way give rise to the Nash solution or solutions. We think that the mere multitude of these models strengthens our feeling that the Nash solution(s) has (have) great theoretical appeal. We apologize already here to those authors whose work has not been mentioned; most probably some works have escaped our attention. For instance, we know that a great deal more can be said about Rubinstein's work and related results (see section 7). Also, we did not mention the work by Aumann and Kurz (1977) who give an account for nonsymmetry of Nash solutions in terms of "boldness" and "fear of ruin".

Our motivation for reviewing all these models has not only been the theoretical one mentioned above. We also hope to have contributed to the applicability of the Nash solution(s) by collecting all these different models.

A brief remark on the n-person case : most models and results have been stated for 2-person games, but in many though not all cases extensions to n-person games are straightforward. The 2-person case has received much more attention in literature (cf. section 2). Some references to work on "n-person Nash solutions" are Lensberg (1982), and Peters (1986b, section 28).

Ongoing research in this field is devoted to finding relations between several kinds of models. In particular, a lot of work can be done on finding strategic models to support axiomatic approaches, and vice versa. Our leasthope is to have provided a list of references to the interested researcher.

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CHAPTER XIII

SOME SOCIAL CHOICE PROBLEMS

by Ton Storcken *)

1. INTRODUCTION

We are all familiar with situations in which a group of individuals has to agree upon a collective choice or preference, e.g. the election of a president or chairman, or the ranking of true-bred dogs by a jury at a dogshow. In fact everyone has been in such situations and probably has become engaged by arguments and the points of view of the other group members brought about in a discussion, which often precedes the decision of the group. Although a great deal of consideration is absorbed by such discussions, whenever we have to make a collective choice, the methods which yield these choices are seldom at stake. In the theory of social choice this is the other way around. There the whole interest is dedicated to the methods, by which groups of individuals make their collective choice or preference and no attention is paid to an eventually preceding discussion.

Otherwise stated, in social choice theory constitutional decision procedures, by which a group of individuals determines its collective choice or preference, are investigated. It is assumed that these procedures depend on the individual preferences over the alternatives among which a group has to decide. Especially the properties of those procedures are studied. It appears that several, at first sight socially defendable and appreciable conditions for those procedures are contradicting each other, whenever there are three or more alternatives. These results are often called impossibility theorems and play an important role in social choice theory. For this reason this chapter is structured by these impossibility theorems.

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The structure of this chapter is as follows :

In section 2 the notion decision procedure is formalized and illustrated by some (well-known) examples. In section 3 several conditions for these procedures are discussed. In the next section a short historical survey of social choice theory is given and finally in section 5 a new impossibility theorem is discussed, in order to make the reader acquainted with the technical aspects of the impossibility theorems.

It is notified here that this chapter on social choice theory does not give a "total" or "complete fractional" survey of social choice theory. This has been done already, even lately, by e.g. Sen (1986) in which the reader may find many references to that theory. This chapter tries to introduce the reader to the problems which are studied by social choice theory.

2. WELFARE FUNCTIONS AND CHOICE FUNCTIONS

In this section we introduce two models by which constitutional decision rules can be studied. Both models use the following notions :

$A = \{a_1, a_2, \ldots, a_p\}$ called the set of *alternatives*. Hence there are $p = |A|$ alternatives. They are often indicated by small letters : a,b,c, x,y, and z.

$N = \{1,2,\ldots,n\}$ called the set of *individuals*. There are $2 \leqq n = |N|$ individuals, which are frequently indicated by small letters : i,j,k,l, m and n.

Let R be a relation on A, i.e. $R \subset A \times A$. Then

$\bar{c}R := \{\langle x,y\rangle \in A \times A : \langle x,y\rangle \notin R\}$ is the *complement* of R

$\bar{a}R := \{\langle x,y\rangle \in R : \langle y,x\rangle \notin R\}$ is the *asymmetric part* of R

$\bar{s}R := \{\langle x,y\rangle \in R : \langle y,x\rangle \in R\}$ is the *symmetric part* of R

$\bar{d}R := \{\langle x,y\rangle \in R : x = Y\}$ is the *diagonal part* of R

$\bar{v}R := \{\langle x,y\rangle \in A \times A : \langle y,x\rangle \in R\}$ is the *converse* of R

$R_\circ R := \{\langle x,z\rangle \in A \times A : \exists y \in A \; \langle x,y\rangle \in R \,\&\, \langle y,z\rangle \in R\}$ is the *composition* of R with itself

Furthermore R is called

transitive if $(R_o R) \subset R$

negatively transitive if $\bar{c}R$ is transitive

asymmetric if $\bar{s}R = \emptyset$

irreflexive if $\bar{d}R = \emptyset$

complete if $R \cup (\bar{v}R) \cup \bar{d}\bar{c}\bar{d}R = A \times A$

a *linear ordering* if it is transitive, asymmetric and complete. The set of linear orderings on A is indicated by $\mathbb{L}(A)$

a *weak ordering* if it is negatively transitive and asymmetric. The set of weak orderings on A is denoted by $\mathbb{W}(A)$.

a *quasi-ordering* on A if it is transitive and asymmetric. The set of quasi-orderings on A is denoted by $\mathbb{Q}(A)$.

A *society* Γ is an ordered pair $\langle A,N \rangle$, where A is a set of alternatives among which the individuals of set N have to make a collective choice.

Let $\Gamma = \langle A,N \rangle$ be a society and V a set of orderings on A, e.g. $V = \mathbb{W}(A)$. Then V^n is the n-fold cartesian product set of V. Let $r \in V^n$ such that $r = \langle R_1, R_2, \ldots, R_n \rangle$, then r represents a combination of individual orderings, where R_i is the preference ordering of individual i at combination r. r is called a *profile*. The set of profiles V^n is the set of possible combinations of individual orderings and is the domain of the decision rules in which we are interested.

EXAMPLE 2.0. Let $A = \{x,y,z\}$, $N = \{1,2\}$, $xyz : R^1$ is the representation of $\{\langle x,y \rangle, \langle y,z \rangle, \langle x,z \rangle\}$, $x\binom{y}{z} : R^2$ is the representation of $\{\langle x,y \rangle, \langle x,z \rangle\}$ and $\binom{xy}{z} : R^3$ is the representation of $\{\langle x,y \rangle\}$. Then x is preferred to y in R^1, y and z are incomparable in R^2, $R^1 \in \mathbb{L}(A)$, $R^2 \in \mathbb{W}(A) - \mathbb{L}(A)$ and $R^3 \in \mathbb{Q}(A) - \mathbb{W}(A)$. It is easy to prove that $\mathbb{L}(A) \subset \mathbb{W}(A) \subset \mathbb{Q}(A)$. Furthermore $(R^2,R^1) \in \mathbb{W}(A)^2$ is a profile, where R^2 is the ordering of individual 1 and R^1 is the ordering of individual 2.

DEFINITION 2.1. *Welfare function / Choice function.* Let $\Gamma = \langle A,N \rangle$ be a society and V a set of orderings on A. Then :

F is a *welfare function* from V^n to W (on Γ) if W is a set of orderings on A and F is a function from V^n to W.

K is a *choice function* from V^n to A (on Γ) if K is a function from V^n to $\mathbb{P}(A)$, the power set of A.

A welfare function F from V^n to W assigns to every possible combination of individual orderings r in V^n an ordering F(r) in W, which is interpreted as the collective or aggregated preference of society Γ at situation r. Similar for a choice function K, K(r) is interpreted as the choice of society at situation r. It is evident that these functions describe constitutional decision rules, which depend only on the individual preferences in V.

Unless otherwise stated, we assume that V and W are both the set of linear orderings on A and by a welfare function (choice function) on $\Gamma = \langle A,N \rangle$, we mean a welfare function (choice function) from $\mathbb{L}(A)^n$ to $\mathbb{L}(A)(A)$ on Γ.

Notice that every welfare function F from V^n to W on Γ induces a corresponding choice function K_F on Γ in the following way :

$\forall r \in \mathbb{L}(A)^n \; K_F(r) := Max(R)$,

where $Max(R) = \{a \in A : \forall b \in A \langle b,a \rangle \notin R\}$

Max(R) is the set of maximal elements of R.

If $R \in \mathbb{Q}(A)$, then $Max(R) \neq \emptyset$. This follows by the transitivity of R and the finiteness of A. For $R \in \mathbb{L}(A)$ it follows by the completeness of R that $|Max(R)| = 1$.

Now several welfare and choice functions are discussed.

EXAMPLE 2.2. *Pairwise majority rule.* Let $\Gamma = \langle A,N \rangle$ be a society. The pairwise majority rule on Γ is a welfare function F from $\mathbb{L}(A)^n$ to $\mathbb{P}(A \times A)$ (the power set of $A \times A$) on Γ defined as follows :

$\forall \langle R_1, R_2, \ldots, R_n \rangle = r \in \mathbb{L}(A)^n, \forall \langle a,b \rangle \in A \times A$

$\langle a,b \rangle \in F(r) :\Leftrightarrow |\{i \in N : \langle a,b \rangle \in R_i\}| > |\{j \in N, \langle b,a \rangle \in R_j\}|$.

Evidently F is well-defined. Note that $|\{i \in N : \langle a,b \rangle \in R_i\}|$ is the number of individuals who prefer a to b. Hence $\langle a,b \rangle \in F(r)$ if and only if there are more individuals who prefer a to b than individuals who prefer b to a at combination r. This explains the name pairwise majority rule. If $|A| = p = 2$, then the range of F is equal to $\mathbb{W}(A)$. If moreover, $|N| = n$ is odd the range is $\mathbb{L}(A)$. May (1952) investigated the rule C from $\mathbb{L}(A)^n$ to $\mathbb{P}(A)$, where $C(r) = Max(F(r))$ for the case where

$|A| = 2$. He has found a nice characterization of this rule, which evidently is strongly related to $F : C = K_F$.
If $|N| = 2$, then the range of F is $\mathfrak{Q}(A)$. Unfortunately the range of F contains cyclical relations, whenever $|N| \geqq 3$ and $|A| \geqq 3$. We will illustrate this fact.

Without loss of generalization suppose $|A| = 3$ and $A = \{a,b,c\}$. Since $N \geqq 3$ there are n_1, n_2 and n_3 such that $n_1 \geqq 1$, $n_2 \geqq 1$, $n_3 \geqq 1$, $n_1+n_2+n_3 = n$ and for all $\{i,j,k\} = \{1,2,3\} : n_i+n_j > n_k$.
Take the following profile r as follows :

$a\ b\ c : R_i$ for $1 \leqq i \leqq n_1$

$b\ c\ a : R_i$ for $n_1 < i \leqq n_1+n_2$

$c\ a\ b : R_i$ for $n_1+n_2 < i \leqq n_1+n_2+n_3 = n$.

Notice that $|\{i \in N : \langle a,b\rangle \in R_i\}| = |\{i \in N : i \leqq n_1 \text{ or } n_1+n_2 < i\}|$

$$= n_1+n_3 > |\{i \in N : \langle b,a\rangle \in R_i\}| .$$

Hence there are more individuals preferring a to b, than individuals who prefer b to a. Therefore $\langle a,b\rangle \in F(r)$ and $\langle b,a\rangle \notin F(r)$. Similarly it follows $\langle b,c\rangle \in F(r)$, $\langle c,b\rangle \notin F(r)$, $\langle c,a\rangle \in F(r)$ and $\langle a,c\rangle \notin F(r)$. Thus F(r) is cyclical.

Although F is a "nice" and often used rule for societies with a two alternative set, it is almost useless for societies with more than two alternatives.

The profile described above is usually called Condorcet profile named after Condorcet, who studied the probability of the appearance of such a profile. (See Condorcet (1785)). The alternatives in C(r) are frequently called Condorcet-winners.

EXAMPLE 2.3. *Scoring rules*. These rules are characterized by Young (1975). Let $\Gamma = \langle A,N\rangle$ be a society, and $s = \langle s_1,s_2,\ldots,s_p\rangle \in \mathbb{R}^p$ such that $s_1 \geqq s_2 \geqq \ldots \geqq s_p \geqq 0$; s is called a *score vector*. Let $r \in \mathbb{W}(A)^n$ be a profile then every individual i assigns scores to every alternative x, according to its preference R_i. First we calculate the number of alternatives defeated by x in $R_i : d^-(x,R_i) := |\{y \in A : \langle x,y\rangle \in R_i\}|$. Then we calculate the number of alternatives which defeat x in $R_i : d^+(x,R_i) :=$ $|\{y \in A : \langle y,x\rangle \in R_i\}| = d^-(x,\bar{v}R_i)$. Let $d^0(x,R_i) = |A| - d^+(x,R_i) -$

$d^-(x,R_i)$. $d^0(x,R_i)$ is the number of alternatives, which are incomparable to x in R_i. Hence if R_i is a linear ordering $d^0(x,R_i) = 1$.

Now the score of x at R_i can be computed :

$$\text{score}(x,R_i,s) := \frac{1}{d^0(x,R_i)} \cdot \sum_{d^+(x,R_i) < t \leqq (p - d^-(x,R_i))} s_t .$$

For instance if $s = \langle 3,2,1 \rangle$ and $a\binom{b}{c} : R^2$, then $\text{score}(b,R^2,s) := \frac{1}{2} \cdot \sum_{1 < t \leqq (3-0)} s_t = \frac{1}{2}(s_2+s_3) = 1\frac{1}{2}$.

The score of x at a profile r is equal to the sum of all the individual scores :

$$\text{score}(x,r,s) := \sum_{i \in N} \text{score}(x,R_i,s).$$

Now the society ranks the alternatives according to their score :

The welfare scoring function F_s with score vector s is the welfare function F_s from $\mathbb{W}(A)^n$ to $\mathbb{W}(A)$ on Γ defined as follows :

$\forall r \in \mathbb{W}(A)^n \ \ \forall a,b \in A \ \ \langle a,b \rangle \in F_s(r) :\Leftrightarrow \text{score}(a,r,s) > \text{score}(b,r,s)$.

Evidently F_s is well-defined.

The choice scoring function K_s with score vector s is the choice function K_{F_s} induced by F_s.

Hence $x \in K_s(r)$ if and only if $\forall y \in A \ \ \text{score}(x,r,s) \geqq \text{score}(y,r,s)$.

There are two special applications of choice scoring functions, both corresponding with a special choice of s. The first one is the Borda rule. This rule has been introduced by Borda (1781). It is defined as follows :

The *Borda welfare function* BW is the welfare scoring function F_s from $\mathbb{W}(A)^n$ to $\mathbb{W}(A)$ on Γ with score vector $s = \langle p,p-1,\ldots,2,1 \rangle$.

The *Borda choice function* BC is the choice scoring function K_s from $\mathbb{W}(A)^n$ to $\mathbb{P}(A)$ on Γ with score vector $s = \langle p,p-1,\ldots,2,1 \rangle$.

We are very familiar with the second application namely : The ordinary voting rule (one man one vote).

The *voting rule* V is the choice scoring function K_s from $\mathbb{L}(A)^n$ to $\mathbb{P}(A)$ on Γ, with score vector $\langle 1,0,0,\ldots,0,0 \rangle$.

Since only the maximal elements of an individual preference R_i of combination r score, it is evident that V is indeed the "one man one vote" rule.

Let us illustrate these two types of applications with some examples.

Suppose $N = \{1,2,\ldots,75\}$ and $A = \{a,b,c,d,e\}$. Observe the following profile $r \in \mathbb{L}(A)^{75}$:

$$
\begin{array}{ll}
a\ b\ c\ d\ e : R_i & \text{for } 1 \leqq i \leqq 16 \\
d\ b\ c\ e\ a : R_i & \text{for } 17 \leqq i \leqq 31 \\
c\ b\ e\ d\ a : R_i & \text{for } 32 \leqq i \leqq 46 \\
b\ c\ e\ d\ a : R_i & \text{for } 47 \leqq i \leqq 61 \\
e\ b\ c\ d\ a : R_i & \text{for } 62 \leqq i \leqq 75.
\end{array}
\tag{2.3.1}
$$

Take $s_1 = \langle 1,0,0,0,0 \rangle$. It is easy to compute the following results : score $(a,r,s_1) = 16$, score $(e,r,s_1) = 14$ and score $(d,r,s_1) =$ score $(c,r,s) =$ score $(b,r,s_1) = 15$. Hence the alternative chosen by the voting rule V is a. This however is not an acceptable choice for most of the voters, since 59 ($\approx 4/5$ part) of the voters rank a as worst alternative, whereas b is ranked best or second best by all the voters. Therefore b would be a more acceptable choice in profile (2.3.1). It is this criticism which stimulated de Borda to create the Borda rules as defined above. The result $V(r) = \{a\}$ instead of the choice b occurs because not all the positions of the alternatives are taken into account in the voting rule V. If we take $s_2 = \langle 5,4,3,2,1 \rangle$ it is straightforward to prove that score$(a,r,s_2) =$ 139, score$(b,r,s_2) = 315$, score$(c,r,s_2) = 270$, score$(d,r,s_2) = 195$ and score$(e,r,s_2) = 206$. Hence b c e d a : BW(r) and $\{b\} = BC(r)$.

Although these rules give an acceptable result in profile (2.3.1), they do not always yield a nice result.

Observe the following two profiles :

$$
\begin{array}{lll}
r : & a\ b\ e\ c\ d : & R_1 \\
& a\ e\ b\ c\ d : & R_i \text{ for } 2 \leqq i \leqq 35 \\
& e\ a\ b\ c\ d : & R_i \text{ for } 36 \leqq i \leqq 74 \text{ and} \\
& a\ b\ e\ d\ c : & R_{75}.
\end{array}
\tag{2.3.2}
$$

$$
\begin{array}{lll}
r' : & a\ b\ c\ d\ e : & R'_1 \\
& \text{For all } i \geqq 2 : & R'_i = R_i.
\end{array}
\tag{2.3.3}
$$

Profile r' and r differ only in the first component. Again it is straightforward to calculate :

$\text{score}(a,r,s_2) = \text{score}(a,r',s_2) = 336,$

$\text{score}(b,r,s_2) = \text{score}(b,r',s_2) = 227,$

$\text{score}(c,r,s_2) = 149$ and $\text{score}(c,r',s_2) = 150$

$\text{score}(d,r,s_2) = 76$ and $\text{score}(d,r',s_2) = 77$ and

$\text{score}(e,r,s_2) = 337$ and $\text{score}(e,r',s_2) = 335.$

Hence BC(r) = e and BC(r') = a. Looking at the two profiles we observe that BC(r') is individual 1's best choice in R_1. Hence if R_1 is his real preference he may take advantage of the situation at (2.3.2) by misrepresenting his real preference by R_1'.

A, to the author's opinion, more illustrative defect of the Borda rule can be found in the following example.

Suppose N = {1,2,3,4} and A = {a,b,c,d,e}.

Observe the following two profiles

r^2 a b c d e : R_1^2 (2.3.4)

a b c e d : R_2^2

e d c $\binom{a}{b}$: R_3^2

e d c b a : R_4^2

and

r^3 a b c d e : R_1^3 (2.3.5)

b a c e d : R_2^3

d e c $\binom{a}{b}$: R_3^3

d e c b a : R_4^3.

R_2^3 can be constructed by a positional switch of the pair {a,b} in R_2^2. Moreover R_2^2 and R_2^3 differ only on the pair {a,b}. Notice that R_3^2 and R_3^3 differ only on the pair {d,e} and R_4^3 and R_4^2 too. Hence r^2 and r^3 differ only on three pairs of alternatives.

Note : $\text{score}(a,r^2,s_2) = 12\tfrac{1}{2}$, $\text{score}(a,r^3,s_2) = 11\tfrac{1}{2}$,

$\text{score}(b,r^2,s_2) = 11\tfrac{1}{2}$, $\text{score}(b,r^3,s_2) = 12\tfrac{1}{2}$

$\text{score}(c,r^2,s_2) = 12$, $\text{score}(c,r^3,s_2) = 12$

$\text{score}(d,r^2,s_2) = 11$, $\text{score}(d,r^3,s_2) = 13$

$score(e,r^2,s_2) = 13$, $score(e,r^3,s_2) = 11$.

Hence e a c b d : $BW(r^2)$ and d b c a e : $BW(r^3)$.

Hence $\bar{v}BW(r^2) = BW(r^3)$. $BW(r^3)$ is the totally opposite ordering of $BW(r^2)$. Hence $BW(r^3)$ and $BW(r^2)$ differ on every pair. Hence to construct $BW(r^3)$ from $BW(r^2)$ we have to change $\binom{p}{2} = \binom{5}{2} = 10$ pairs, but as is shown above we can construct r^3 from r^2 by only changing 3 pairs of alternatives in the individual orderings. Hence a "small change" in the profile may cause a "great change" in the corresponding Borda welfare outcomes.

EXAMPLE 2.4. *Voting by veto*. We will not introduce this rule in a formal way but illustrate it with a few examples.

Suppose : $A = \{a_1,a_2,a_3,a_4,a_5,a_6,a_7\}$ and
$N = \{1,2,3\}$.

Now the veto voting rule is applied as follows.

First individual 1 picks an alternative from the set of possible outcomes. This alternative will not be chosen. Then individual 2 takes an alternative from the remaining set of possible outcomes. Then 3 does the same and after 3 it is 1's turn again. This procedure continues until there is only one alternative left. This is the choice of the veto voting functions.

For example take the following profile :

$$\begin{array}{ll} r & a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7 \ :\ R_1 \\ & a_3\ a_5\ a_7\ a_4\ a_6\ a_2\ a_1 \ :\ R_2 \\ & a_1\ a_2\ a_6\ a_4\ a_7\ a_5\ a_3 \ :\ R_3. \end{array} \tag{2.4.1}$$

After 1 has removed a candidate, the set of possible outcomes is reduced to $\{a_1,a_2,a_3,a_4,a_5,a_6\}$. After 2's turn this set is $\{a_2,a_3,a_4,a_5,a_6\}$. After 3's turn this set is $\{a_2,a_4,a_5,a_6\}$. Then the set is reduced to $\{a_2,a_4,a_5\}$ by 1, and furthermore to $\{a_4,a_5\}$ by 2 and finally 3 reduces it to $\{a_4\}$ which is the choice of the veto voting function at profile r.

Notice that a_4 is more or less acceptable for every individual.

Now take the following profile

$$\begin{array}{ll} r' & a_1\ a_2\ a_3\ a_5\ a_4\ a_6\ a_7 \ :\ R_1' \\ & a_4\ a_3\ a_5\ a_7\ a_2\ a_3\ a_6 \ :\ R_2' \\ & a_4\ a_1\ a_2\ a_6\ a_7\ a_5\ a_3 \ :\ R_3' \end{array} \tag{2.4.2}$$

Applying this rule to profile r' we obtain subsequently the following sets of possible outcomes : $\{a_1,a_2,a_3,a_4,a_5,a_6\}$, $\{a_1,a_2,a_3,a_4,a_5\}$, $\{a_1,a_2,a_4,a_5\}$, $\{a_1,a_2,a_5\}$, $\{a_1,a_5\}$ and $\{a_1\}$. Hence the veto voting function gives $\{a_1\}$ at profile r^3. Observing the positions of a_4 in r and r', we see that the position of a_4 in r' is an improvement of the position of a_4 in r. However at r' a_4 is not chosen although its position there is better than that in r, where a_4 is chosen. Hence the veto voting function lacks this kind of a monotonicity property.

These veto voting rules are discussed in literature (see, e.g. Moulin (1983)) in connection with a game-theoretical approach to social choice theory.

EXAMPLE 2.5. *Artificial decision rules.* The welfare functions and choice functions introduced above are all used (some in a modified form) in practice. The rules introduced here are not used in daily life or at least we hope that they are not applied. These rules have an artificial character. They are introduced for illustration purposes.

Suppose : $N = \{1,2,\ldots,n\}$ and $A = \{a_1,a_2,\ldots,a_p\}$.

The *dictatorial welfare function* D_i from $\mathbb{L}(A)^n$ to $\mathbb{L}(A)$ on $\Gamma = \langle A,N\rangle$ with *dictator* i is defined as follows :

$$\forall r \in \mathbb{L}(A)^n \quad D_i(r) = R_i.$$
$$r = \langle R_1,\ldots,R_n\rangle$$

Mathematically D_i is the projection of $\mathbb{L}(A)^n$ on the i^{th} coordinate. Since the collective preference is completely determined by i^{th} individual preference, i is decisive in every situation. Hence i is a dictator in rule D_i. Similarly we can define a dictatorial choice function D_i' :

$$D_i' = K_{D_i}.$$

The *antidictatorial welfare function* AD_i from $\mathbb{L}(A)^n$ to $\mathbb{L}(A)$ on $\Gamma = \langle A,N\rangle$ with *antidictator* i is defined as follows :

$$\forall\ r \in \mathbb{L}(A)^n \quad AD_i(r) = \bar{v}R_i.$$
$$r = \langle R_1,\ldots,R_n\rangle$$

Again the i^{th} component in a profile completely determines the images of AD_i, but now these images completely conflict with the i^{th} preference. Therefore i is called an antidictator. Again an antidictatorial choice function AD_i' can be defined by $AD_i' = K_{AD_i}$.

The *constant welfare function* C_R (for some fixed $R \in \mathbb{L}(A)$) from $\mathbb{L}(A)^n$ to $\mathbb{L}(A)$ on Γ is defined as follows :

$\forall r \in \mathbb{L}(A)^n \quad C_R(r) = R.$

The name of this rule speaks for itself. A constant choice function C'_R can be defined as follows : $C'_R = K_{C_R}$.

Of course there are many other interesting rules (see e.g. Moulin (1983)), which we have to omit in view of the available space.

3. POSSIBLE CONDITIONS FOR DECISION RULES

In the previous section several welfare functions and choice functions are introduced. Along with their introduction we have shown some properties of these functions. In this section these properties are formalized. The formalizations discussed here are frequently used as reasonable conditions.

Suppose : $A = \{a_1,\ldots,a_p\}$ and $N = \{a,\ldots,n\}$.

Let further F be a welfare function from V^n to W on $\Gamma = \langle A,N\rangle$ and K be a choice function from V^n to $\mathbb{P}(A)$ on Γ.

DEFINITION 3.1. *Pareto-optimality*.

F is *Pareto-optimal* if $\forall r \in V^n[\forall i,j \in N \quad R_i = R_j] \Rightarrow F(r) = R_1 (= R_2 \ldots = R_n)$

K is *Pareto-optimal* if $\forall r \in V^n[\forall i,j \in N \quad R_i = R_j] \Rightarrow K(r) = Max(R_1)$.

First it should be noted that Pareto-optimality appears in several other forms in literature. Here only one form (the weakest) is discussed to avoid confusion. Many of the conditions introduced hereafter have small variations in literature. We will only discuss one form of each, since they all cover the same intuitive notion in which we are interested.

A decision rule is Pareto-optimal if the whole society is decisive in those situations in which all individuals agree with each other. Clearly the antidictatorial rules and the constant rule of 2.5 are not Pareto-optimal, but all the other rules have this property.

DEFINITION 3.2. *Non-dictatorship.*

F is *non-dictatorial* if

$\forall i \in N \; \exists r \in V^n \quad a\bar{R}_i \not\subset F(r)$.

K is *non-dictatorial* if

$\forall i \in N \; \exists r \in V^n \quad Max(R_i) \not\subset K(r)$.

A decision rule is non-dictatorial if for every individual i there is a situation in which the rule does not adopt i^{th} strict preference. It is evident that the dictatorial rules in 2.5 do not have this non-dictatorship property, but all the others introduced in §2 have.

DEFINITION 3.3. *Independence of Irrelevant Alternatives* (IIA).

F is *independent of irrelevant alternatives* if

$\forall a,b \in A \quad \forall r^1,r^2 \in V^n \quad r^1|_{\{a,b\}} = r^2|_{\{a,b\}} \rightarrow F(r^1)|_{\{a,b\}} = F(r^2)|_{\{a,b\}}$,

where for all $R \in \mathbb{P}(A \times A)$, $R|_{\{a,b\}}$ is the restriction of R to the pair $\{a,b\}$ and for all $r \in V^n$, $r|_{\{a,b\}} = \langle R_1|_{\{a,b\}}, \ldots, R_n|_{\{a,b\}}\rangle$.

A welfare function F is independent of irrelevant alternatives if the collective preference F(r) between any pair of alternatives {a,b} and in any situation $r \in V^n$ does not depend on the position of any other (irrelevant) alternative c($\notin$ {a,b}) in that situation r. Hence the collective preference between a and b is completely determined by the individual preference between a and b. IIA guarantees that the collective preference is established by pairwise comparison of the alternatives.

Note that by the definitions of the majority rule (2.2), the dictatorial, the antidictatorial and the constant welfare functions (2.5), all the functions have property IIA. The Borda welfare function (2.3) is not independent of irrelevant alternatives. In order to prove this, consider the profiles r^2 in (2.3.4) and r^3 in (2.3.5) and observe $r^2|_{\{a,c\}} = r^3|_{\{a,c\}}$ and $BW(r^2)|_{\{a,c\}} \neq BW(r^3)|_{\{a,c\}}$.

In general welfare scoring functions do not have property IIA, unless $|A| = 2$ in which case there are are no irrelevant alternatives.

DEFINITION 3.4. *Non-manipulable.*
K is *non-manipulable* if
$\forall i \in N \quad \forall r^1, r^2 \in V^n \; [\forall j \in N - \{i\} \; R^1_j = R^2_j \Rightarrow$
$[\exists \; a \in K(r^1) - K(r^2), \; b \in K(r^2) - K(r^1) \; \langle a,b \rangle \in \bar{a}R^1_i]$

K is non-manipulable if for all individuals i it holds that for situations r^1 it is not completely profitable for i to change its preference from R^1_i to R^2_i, where all the others do not change their preferences. In 2.3 we showed that the Borda choice function is manipulable. (See profiles (2.3.2) and (2.3.3)). In general, every choice scoring function is manipulable whenever $|A| \geqq 3$. It is left to the reader to show that the antidictatorial and veto choice functions are manipulable. The other rules introduced in 2.2 and 2.5 are non-manipulable. The proof is again left to the reader.

DEFINITION 3.5. *Monotony.*
Let $a,b \in A$, $R^1, R^2 \in \mathbb{P}(A \times A)$ and $r^1, r^2 \in V^n$. The preference $\langle a,b \rangle$ does *not decrease* going from R^2 to R^1 if $[\langle a,b \rangle \in R^2 \Rightarrow \langle a,b \rangle \in R^1]$ and
$[\langle a,b \rangle \in \bar{a}R^2 \Rightarrow \langle a,b \rangle \in \bar{a}R^1]$.
Notation : $\langle R^1, R^2 \rangle \in \Delta(\langle a,b \rangle)$.

The preference $\langle a,b \rangle$ does *not decrease* going from r^2 to r^1 if $\forall i \in N \; \langle R^1_i, R^2_i \rangle \in \Delta(\langle a,b \rangle)$.
Notation :
$\langle r^1, r^2 \rangle \in \Delta_n(\langle a,b \rangle)$.
F is *monotonous* if
$\forall a,b \in A \quad \forall r^1, r^2 \in V^n \quad \langle r^1, r^2 \rangle \in \Delta_n(\langle a,b \rangle) \Rightarrow \langle F(r^1), F(r^2) \rangle \in \Delta(\langle a,b \rangle)$.
K is *monotonous* if
$\forall a \in A \quad \forall r^1, r^2 \in V^n \; [[\forall b \in A \;\; \langle r^1, r^2 \rangle \in \Delta_n(\langle a,b \rangle)]$ and
$[a \in K(r^2)]] \Rightarrow [a \in K(r^1)]$.

Both these types of monotony are also called strong positive association. Since $\Delta_n(\langle a,b \rangle)$ and $\Delta(\langle a,b \rangle)$ are relations on V^n and W respectively, a monotonous welfare function does not disturb these relations. This explains the name monotony. It is easy to prove that a monotonous welfare function is independent of irrelevant alternatives. The anti-dictatorial welfare function however is not monotonic. (The proof is left to the reader).

The veto voting rule (2.4) is not monotonous : this is shown in example 2.4. Muller and Satterthwaite (1977) showed that non-manipulable choice functions K are monotonous, whenever $|K(r)| = 1$ for all $r \in \mathbb{L}(A)^n$.

We end this sequence of conditions with a condition introduced by the author (1987).

DEFINITION 3.6. *Continuity*.
Let $R^1, R^2 \in \mathbb{P}(A \times A)$ and $r^1, r^2 \in V^n$.
$d(R^1,R^2) := |R^1 \Delta R^2| \quad (= |(R^1-R^2) \cup (R^2-R^1)|)$
$d_n(r^1,r^2) := |\cup\{R_i^1 \Delta R_i^2 : i \in N\}|$.
F is *continuous* if
$\forall r^1, r^2 \in V^n \quad d_n(r^1,r^2) \geq d(F(r^1),F(r^2))$.

It can be proved that d and d_n are both distance functions. Furthermore there are topological spaces on V^n and W such that the condition :

$\forall c \subset W \quad c$ is open in $W \Rightarrow F^{-1}(c)$ is open in V^n

is equivalent to the condition :

$\forall r^1, r^2 \in V^n \quad d_n(r^1,r^2) \geq d(F(r^1),F(r^2))$.

These facts will be proved in Storcken (198). According to these facts the name continuity is sufficiently explained. In words this condition is as follows :

F is continuous if a change in the profile is not smaller than a change in the corresponding images.

It is shown that the Borda welfare function is not continuous. Furthermore in Storcken (198) it is shown that the IIA condition implies this continuity condition. Hence the other welfare functions introduced in 2.2 and 2.5 are continuous. It can be shown that a welfare scoring function is not continuous whenever $|A| \geq 3$.

Although there are many other conditions imposable on decision rules we will stop here, because we are sufficiently equipped to understand the impossibility results of the next section.

4. SOME IMPOSSIBILITY THEOREMS IN SOCIAL CHOICE

In 1951 K.J. Arrow proved his famous impossibility theorem, which can be found in Arrow (1978). The theorem has been presented in several forms. Here one of these is stated.

THEOREM 4.1. *Arrow's impossibility theorem*
Let $\Gamma = \langle A,N \rangle$ be a society such that $|A| \geqq 3$ and F a welfare function on Γ from $\mathbb{W}(A)^n$ to $\mathbb{W}(A)$. Then F is not simultaneously Pareto-optimal, independent of irrelevant alternatives and non-dictatorial.

A proof of this theorem can be found in e.g. Kelly (1978). This theorem surprises many social choice theorists, since the three conditions Pareto-optimality, IIA and non-dictatorship seem at first sight reasonable for ordinary decision rules and yet they cannot occur in any welfare function simultaneously. This surprise is and was turned into efforts to recover the cause of this impossibility. These efforts gave rise to a considerable amount of articles and books, which nowadays form the theory of social choice. Disregarding several authors such as de Borda, Caritat Marquis de Condorcet, and May, one could say that Theorem 4.1 initiated the theory of social choice.

The rest of this section comments on the efforts mentioned above. These efforts can be divided into several classes. We will only deal with a few of them.

4.2 *Changes of the range of decision rules*

As you can notice in Theorem 4.1 the impossibility is stated for decision rules with range $\mathbb{W}(A)$. Many work was done on studying the effects of a change of the range of the decision rule.

First it is noticed that substituting $\mathbb{P}(A \times A)$ for $\mathbb{W}(A)$ yields a possibility theorem, e.g. the pairwise majority rule (2.2). But this substitution brings in another not yet solved problem : How to handle the cyclic parts of a relation ?

A second substitution, namely A instead of $\mathbb{W}(A)$ introduces similar impossibility theorems. If we substitute A for $\mathbb{W}(A)$ the welfare function becomes a choice function, the images of which are singletons. Gibbard (1973) and Satterthwaite (1975) independently proved the following result.

THEOREM 4.2.1. Let $\Gamma = \langle A,N \rangle$ be a society such that $|A| \geqq 3$. Then there is no choice function K from $\mathbb{W}(A)$ to A which is simultaneously Pareto-optimal, non-dictatorial and non-manipulable.

Some years later Muller and Satterthwaite (1977) proved an other impossibility theorem of this kind.

THEOREM 4.2.2. Let $\Gamma = \langle A,N \rangle$ be a society such that $|A| \geqq 3$. Then there is no choice function K from $\mathbb{W}(A)$ to A which is simultaneously Pareto-optimal, non-dictatorial and monotonous.

Furthermore they showed that 4.2.2 implies both 4.2.1 and 4.1.

In both these theorems the IIA-condition is replaced by another condition. This fact is motivated by the range changement of the decision rule. Since Theorem 4.2.2 implies Theorem 4.1, monotony is perhaps a weaker condition than IIA.

Blair and Pollack (1979) and Blau (1979) studied welfare functions of which the ranges were still orderings, e.g. $\mathbb{Q}(A)$. They found impossibility results similar to Theorem 4.1. In their theorems "non-dictatorial" is replaced by "non-oligarchical", which means that there is not a group of individuals that is dictatorial as a group.

There are many other impossibility theorems in which the range of the decision rule is changed. Most of these rules are treated in Kelly (1978). All the ranges which are discussed lead to unsatisfactory decision rules or new impossibility theorems.

4.3 *Introduction of infinite societies*

Up to now the societies had finitely many alternatives and finitely many individuals. There are, however, models with either an infinite set of individuals or an infinite set of alternatives. To both types of models we will devote a few words.

Kirman and Sondermann (1972) studied societies with an infinite number of individuals. They introduced a measure on this individual set and translated the Arrow-conditions of Theorem 4.1 to this new model. They found a result which is similar to 4.1, which means that the absence of a dictator is replaced by the absence of a coalition with measure zero, which is decisive in every situation. Noting that singletons of the set of individuals have measure zero, their result is comparable to Arrow's impossibility.

Chichilnisky and Heal (1983) studied models with infinitely many alternatives. In their model IIA is replaced by a continuity property and non-dictatorship by anonimity. The anonimity criterion guarantees similar rights for different individuals in similar situations. It is therefore slightly stronger than the non-dictatorship criterion. The continuity property seems to be weaker than IIA. It is noted here that these comparisons are purely intuitive since a logical one is impossible, because of the differences in the models. Although they have a condition in a topological description : the space of preferences is contractable which is necessary and sufficient to guarantee the existence of a Pareto-optimal, anonymous and continuous welfare function, it is difficult to interprete their result, just because of the topological nature of that condition. Hence one could be doubtful about the use of such a model in order to understand Theorem 4.1.

4.4 *Changes of the conditions of a welfare function*

It is possible to argue about the conditions imposed by Arrow on the welfare functions. One could claim that these are too restrictive and that this is the reason for the occurrence of Arrow's impossibility.
In literature this has been one of the important rejective arguments of those who doubt the use of social choice theory. If we think that Arrow's conditions might be too restrictive, then we are exposed to a new and perhaps even more difficult problem : Is it possible to substitute other conditions for these three, such that the substitutes have a meaningful interpretation and are still powerfull enough to exclude several of the

oddities discussed in section 2.

Many substitutes for the conditions mentioned in section 3 have been investigated. They all caused new impossibilities. We will not deal with these new conditions, for those who like to read more about these, will find a considerable number of these conditions in Kelly (1978), Moulin (1983), Sen (1970) and Pattanaik (1978). We will make one exception. In Storcken (198) it is proved that there is no welfare function on a society with at least three alternatives from $\mathbb{L}(A)^n$ to $\mathbb{W}(A)$, which is simultaneously Pareto-optimal, non-dictatorial and continuous. Since continuity is a weaker condition than IIA, it follows that the above mentioned assertion is a strengthening of Theorem 4.1. Moreover, this indicates that the above stated problem is a difficult one. In section 5 we will deduce a similar result to show how the theorem in Storcken (198) is established.

4.5 *Domain restrictions*

Instead of weakening the three conditions of Theorem 4.1 one could implicitly weaken them by restricting the domain of the welfare function. In an unrestricted domain every possible combination of, say, linear or weak or quasi-orderings is taken into account. This can be interpreted as that the society does not know anything more about the orderings of an individual than that it is linear, weak or a quasi-ordering. Hence domain restrictions can be interpreted as knowledge about the individual orderings.

This type of relaxation of the conditions of Theorem 4.1 led to possibility theorems. Several necessary and sufficient sets of conditions, which indicate special types of domain restrictions, are found in correspondence with the existence of special types of welfare functions. We mentioned already one, namely the theorem of Chichilnisky and Heal (1983). Another well-known result was deduced by Pattanaik and Sen (1969), who characterized the domains which admit pairwise majority rules. Other results on this subject of domain restrictions can be found in Kalai and Muller (1977), Ritz (1985), Storcken (1985) and many, many others.

The problem with these characterizations is that they are very technical and therefore very hard to understand in terms of social choice theory. Again we do not have a clear and satisfying answer to the problem how Theorem 4.1 is brought about.

We end this section by mentioning that there are more types of efforts to understand the cause of Theorem 4.1, e.g. a game theoretical one, see Moulin (1983). None of these efforts, however, gives a clear answer to the posed problem.

5. AN ILLUSTRATIVE IMPOSSIBILITY

In this section a new impossibility theorem is proved to illustrate some of the techniques which are frequently applied to prove an impossibility theorem.

Throughout this section we suppose :

$A = \{a_1, a_2, \ldots, a_p\}$ and $p \geqq 3$,

$N = \{1,2\}$,

$\Lambda = \{X \subset A : X \neq \emptyset$ and $X \neq A\}$ and

$\Gamma = \langle A, N \rangle$.

Hence Γ is a society with at least three alternatives and precisely two individuals. Λ is the set of non-trivial subsets of A. Λ can be seen as the set of (real) choices which one can make from A.

Let G be a function from $\Lambda \times \Lambda$ to Λ. G can be interpreted as a decision rule on Γ, which assigns to every possible combination of (real) individual choices a (real) collective choice. The author was inspired by Brams and Fishburn (1982) to study these functions.

DEFINITION 5.1.

G is *Pareto-optimal* if

$\forall X \in \Lambda \quad G(X,X) = X$;

G is *dictatorial* if

$\forall X,Y \in \Lambda \quad G(X,Y) = X$ or $\forall X,Y \in \Lambda \; G(X,Y) = Y$;

G is *continuous* if

$\forall X^1,X^2,Y^1,Y^2 \in \Lambda \quad d_2(\langle X^1,Y^1\rangle,\langle X^2,Y^2\rangle) \geqq d(G(X^1,Y^1),G(X^2,Y^2))$

where

$d_2(\langle X^1,Y^1\rangle,\langle X^2,Y^2\rangle) := |(X^1 \Delta X^2) \cup (Y^1 \Delta Y^2)|$ and
$d(X^1,X^2) := |X^1 \Delta X^2|$ for all $X^1,X^2,Y^1,Y^2 \in \Lambda$.

Notice that d and d_2 are distance functions. Hence Definition 5.1 is just a translation of equally named conditions defined in section 2.

THEOREM 5.2. If G is Pareto-optimal and continuous, then G is dictatorial.

This theorem is a new impossibility theorem, since the continuity condition is new (as far as the author knows). The impossibility is similar to Arrow's impossibility.

We will prove this theorem in steps.

Observe that for every $X^1,X^2 \in \Lambda$ and $Y^1,Y^2 \in \Lambda$ it holds that :

$d(X^1,Y^1) \leqq d(X^1,(A-X^1)) = p.$

$d_2(\langle X^1,Y^1\rangle,\langle X^2,Y^2\rangle) \leqq p.$

For every $U \in \Lambda$ define :

$\text{Covers}(d,U,p) := \{Y \in \Lambda : \forall X \in U\ d(X,Y) < p\};$

$\text{diag}(U) = \{\langle X,X\rangle : X \in U\};$

$\text{Covers}(d_2,\text{diag}(U),p) := \{\langle X,Y\rangle \in \Lambda \times \Lambda : \forall \langle Z,Z\rangle \in \text{diag}(U)\ d_2(\langle X,Y\rangle,\langle Z,Z\rangle) < p\};$

If $X,Y \in \Lambda$ $U_{X,Y} := \{Z \in \Lambda : Z \cap X \cap Y \neq \emptyset \text{ or } (A-Z) \cap (A-X) \cap (A-Y) \neq \emptyset\}.$

$\text{Covers}(d,U,p)$ is the subset of Λ whose elements are closer than p to any element of U.

LEMMA 5.3. Let $X,Y \in \Lambda$. Then $\langle X,Y\rangle \in \text{Covers}(d_2,\text{diag}(U_{X,Y}),p)$.

PROOF OF LEMMA 5.3. Let $Z \in U_{X,Y}$ and $X,Y \in \Lambda$.

$d_2(\langle X,Y\rangle,\langle Z,Z\rangle) = |X\Delta Z \cup Z\Delta Y| = |A-([X \cap Y \cap Z] \cup [(A-X) \cap (A-Y) \cap (A-Z)])|$

Hence by the definition of $U_{X,Y}$ it follows that $d_2(\langle X,Y\rangle,\langle Z,Z\rangle) < p$, which completes the proof. □

The following lemma strengthens the Pareto-optimality.

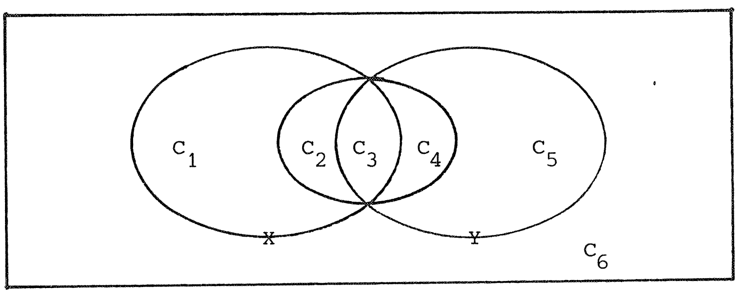
LEMMA 5.4. $\forall X,Y \in \Lambda \quad X \cap Y \subset G(X,Y) \subset X \cup Y$.

PROOF OF LEMMA 5.4. Let $X,Y \in \Lambda$. Take $Z = G(X,Y) \in \Lambda$ and $T \in U_{X,Y}$.
By the Pareto-optimality of G it follows that
$G(T,T) = T$. (5.4.1)
By the continuity of G we have
$d(G\langle T,T\rangle, G\langle X,Y\rangle) \leqq d_2(\langle T,T\rangle,\langle X,Y\rangle)$.
Hence by (5.4.1) and Lemma 5.3 it follows that : $d(T,Z) < p$.
Hence $Z \in$ Covers$(d,U_{X,Y},p)$.
Notice that Covers$(d,U_{X,Y},p) = \{T \in U : (A-T) \notin U\}$.
Hence $(A-Z) \notin U_{X,Y}$.
This leads to $(A-Z) \cap X \cap Y = \emptyset$ and $(A-(A-Z)) \cap (A-X) \cap (A-Y) = \emptyset$.
Hence $X \cap Y \subset Z = G(X,Y) \subset X \cup Y$. □

The following lemma is almost the result we are aiming at.

LEMMA 5.5. Let $X,Y \in \Lambda$ such that $X-Y \neq \emptyset$ and $Y-X \neq \emptyset$.
Then $G(X,Y) \in \{X,Y\}$.

PROOF OF LEMMA 5.5. Let $Z = G(X,Y)$,
$C_2 = (X \cap Z)-Y$, $C_3 = (X \cap Z \cap Y)$, $C_4 = (Y \cap Z)-X$,
$C_1 = X - (Y \cup Z)$, $C_5 = Y - (X \cup Z)$ and $C_6 = A - (X \cup Y)$.
By lemma 5.4 we have $C_3 = X \cap Y \subset Z \subset C_2 \cup C_3 \cup C_4$.



Case 1 Suppose $C_1 \neq \emptyset$ and $C_5 \neq \emptyset$.
Notice $d_2(\langle C_1,C_5\rangle,\langle X,Y\rangle) = | C_2 \cup C_3 \cup C_4 |$.
Hence by the continuity of G we have :

$$d(G(C_1,C_5),Z) \leqq | C_2 \cup C_3 \cup C_4 |. \qquad (5.5.1)$$

By Lemma 5.4 it follows that $G(C_1,C_5) \subset C_1 \cup C_5$.
Hence since $G(C_1,C_5) \neq \emptyset$ we have
$d(G(C_1,C_5),Z) > | C_2 \cup C_3 \cup C_4 |$.
This contradicts (5.5.1).

Case 2 Suppose $C_2 \neq \emptyset$ and $C_4 \neq \emptyset$.
Similarly to case 1 we can deduce a contradiction.
Hence ($C_1 = \emptyset$ or $C_5 = \emptyset$) and ($C_2 = \emptyset$ or $C_4 = \emptyset$).
Using the assumptions $X-Y \neq \emptyset$ and $Y-X \neq \emptyset$ it follows evidently that
$G(X,Y) \in \{X,Y\}$. □

The following result uses the fact that $| A | \geqq 3$.

LEMMA 5.6. $\forall X \in \Lambda \quad G(X,A-X) = X$ or
$\forall X \in \Lambda \quad G(X,A-X) = A-X$.

Since $\langle X,A-X\rangle$ represents a maximal conflict between 1 and 2, this theorem shows that either in every maximal conflict 1 completely wins or in every maximal conflict 2 completely wins. Hence in a maximal conflict there is no intermediate outcome.

PROOF OF LEMMA 5.6. Suppose : $Y \subset X \in \Lambda$ and $X-Y = \{x\}$. By Lemma 5.5 we have $G(X,A-X) \in \{X,A-X\}$. Without loss of generality suppose $G(X,A-X) = X$. It is sufficient to prove that $G(Y,A-Y) = Y$. Again by Lemma 5.5 we have $G(Y,A-Y) \in \{Y,A-Y\}$. Since $d(X,A-Y) \geqq 2$ and $1 = d_2(\langle X,A-X\rangle,\langle Y,A-Y\rangle) \geqq d(G(X,A-X),G(Y,A-Y))$ the proof is finished. □

PROOF OF THEOREM 5.2. By Lemma 5.6 we only have to distinguish the follow-

Case 1 $\forall X \in \Lambda \quad G(X,A-X) = A-X$. We will prove that in this case the following holds : $\forall X,Y \in \Lambda \quad G(X,Y) = Y$. Take $X,Y \in \Lambda$. It is sufficient to prove that $G(X,Y) = Y$. We are done if $X = Y$. Suppose $X \neq Y$. Then we have : $p > d_2(\langle X,Y\rangle,\langle X,A-X\rangle) = d(Y,A-X) \geqq d(G(X,Y),A-X)$ 5.2.1

If $X-Y \neq \emptyset$ and $Y-X \neq \emptyset$, it follows by 5.5, 5.2.1, and the fact $d(X,A-X) = p$, that $G(X,Y) = Y$.

If $X \subset Y$ or $Y \subset X$, it follows by 5.4 and 5.2.1 that $G(X,Y) = Y$.

Case 2 $\forall X \in \Lambda \quad G(X,A-X) = X$.

Similarly to case 1 it follows that : $\forall X,Y \in \Lambda \quad G(X,Y) = X$. □

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